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The Concept and Measurement of Impedance in Periodically Loaded Wave Guides*

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A generalization of ordinary circuit theory which enables one to define impedances in any periodic structure is developed, based on the concept of expansion of electromagnetic fields in terms of a set of linearly independent basis fields. Techniques for measurement of impedances in a periodic structure are described, involving a determination of the parameters of a coupling system by an extension of the well-known nodal shift method.

I. INTRODUCTION

IN a coupling system connecting two different transmission lines or wave guides† it is usually desired that a matched line on the load side be reflected through the coupling system as a match in the input transmission line. The obvious way to test for this is to install a matched load and then measure the input SWR. However, an accurate match may be very difficult to obtain; or if the output transmission line happens to be of some special construction, such as the periodically disk-loaded wave guides used in particle accelerators,[1] there may be no independent method of determining when the load line is matched, other than looking at it through some coupling system.

It is always easy, however, to produce a pure reactance (i.e., completely reflecting) termination of the load line, and if the coupling system is lossless, its input impedance must also be a pure reactance. This reactance is determined merely by locating the position of the extremely sharp node on the input transmission line, so that if we can find a way of determining the coupling system parameters from the relation of load reactance to input reactance, we can expect this method to give us a more convenient as well as more accurate determination of the coupling system performance. Once the parameters of the coupling system are known, the relations between its input and output impedance may be used backwards to measure impedances in the load transmission system.

In Sec. II we review the well-known theory of the nodal shift method as applied to "smooth" transmission lines. Sections III, IV, and V are devoted to developing the concept of impedance in periodically loaded wave guides, and in Sec. VI the theory of the nodal shift method is extended to such guides. The resulting measurement technique was developed and used by the writer in 1946 in connection with the design of coupling systems for the Stanford linear accelerator tubes.[1]

II. NODAL SHIFT THEORY

First, we note the familiar fact that the voltage-current relationships of any linear, passive, four-

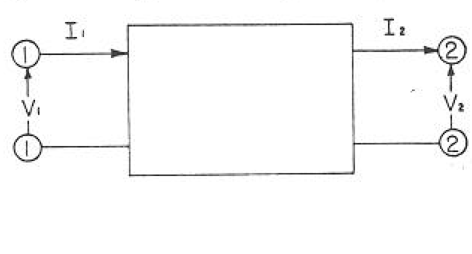

FIG. 1. Voltage and current conventions for a four-terminal network.

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† We will understand the term "transmission line" to mean any structure in which waves may be propagated according to the usual transmission-line equations. The term thus includes wave guides, coaxial lines, and two-wire lines.

[1] W. W. Hansen, *Consiglio Nazionale de Ricerche* (Tipografia del Senato, Roma, 1948), p. 111; Ginzton, Hansen, and Kennedy, Rev. Sci. Instr. **19**, 89 (1948); E. L. Chu and W. W. Hansen, J. Appl. Phys. **18**, 996 (1947); **20**, 280 (1949); E. L. Chu, Ph.D. thesis (Stanford University, 1951).

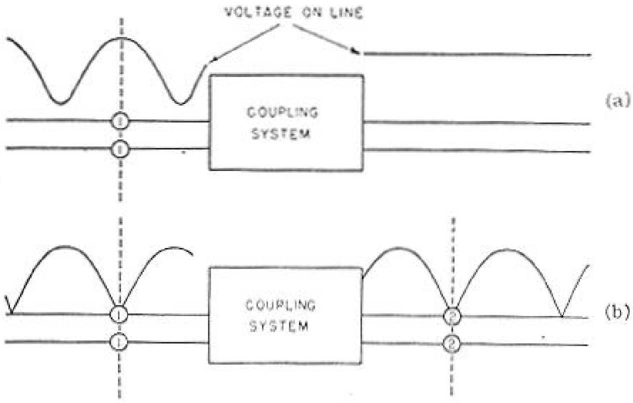

FIG. 2. Definition of reference planes. (a) The load line is matched; (b) the load is a pure reactance.

terminal network may be expressed in terms of the network parameters A, B, C, D, in the form (see Fig. 1 for notation)

$$\begin{aligned} V_1 &= AV_2+BI_2, \\ I_1 &= CV_2+DI_2, \\ \begin{vmatrix} A & B \\ C & D \end{vmatrix} &= 1. \end{aligned} \tag{1}$$

For a lossless network, A and D are real, while B and C are imaginary.[2] The input impedance $Z_1=V_1/I_1$ is thus given in terms of the load impedance $Z_2=V_2/I_2$ by

$$Z_1=(AZ_2+B)/(CZ_2+D). \tag{2}$$

We may consider the network of which the coupling system is a part as including arbitrary lengths of input and output transmission lines. We now show that for a lossless system the reference planes (terminals) may be chosen so as to simplify (2). Assume that voltages and currents are so defined that all impedances are normalized, i.e., $Z=1$ on any matched transmission line; and consider the two situations depicted in Fig. 2. In Fig. 2(a) the output line is matched. Choose terminals (1) at a voltage maximum on the input line. Then the impedance Z_1 is real and greater than unity. This particular value of impedance is numerically equal to the voltage SWR η seen on line 1 when line 2 is matched, and may be called the SWR of the coupling system. From Eq. (2) we have

$$\eta=(A+B)/(C+D).$$

In Fig. 2(b) we have chosen the output terminals so that a short there also places a short at the input terminals. For this case, Eq. (2) reduces to

$$B/D=0, \quad \text{or} \quad B=0.$$

But since η is real, we must have $C=0$ also. Therefore $\eta=A/D$, and the general impedance relation (2) reduces, for this choice of terminals, to

$$Z_1=\eta Z_2. \tag{3}$$

In general, the input impedance to a shorted transmission line of length x is $jZ_0 \tan\theta$, with $\theta=2\pi x/\lambda$ the electrical length of the line. With normalized impedances, $Z_0=1$. From this we see that if a short is placed a distance θ_2 to the right of terminals (2), there will be a short a distance θ_1 to the right of terminals (1) (Fig. 3), where

$$\tan\theta_1=\eta \tan\theta_2. \tag{4}$$

If one places a movable short in the load transmission line and measures the position of the short in the input line as a function of the position of the load short, Eq. (4) will enable one to find from these data the value of η and the positions of the two reference planes. Usually these are the only parameters of the coupling system in which one is interested.

In practice one would plot the difference $(\theta_1-\theta_2)$ as a function of θ_2, or if more convenient, the distances (x_1-x_2) and x_2 may be plotted directly. Since the reference points are not in general known at this stage of the process, one measures position of the nodes from arbitrary origins, and the origin of the $(\theta_1-\theta_2)$ *vs* θ_2 graph remains arbitrary. The shape and orientation of the plot, however, are determined by the value of η. The equation represented is

$$\begin{aligned} \theta_1-\theta_2 &= \tan^{-1}\eta \tan\theta_2-\tan^{-1}\tan\theta_2 \\ &= \tan^{-1}\frac{(\eta-1)\tan\theta_2}{1+\eta\tan^2\theta_2}. \end{aligned} \tag{5}$$

As a function of θ_2 this reaches a maximum of

$$\Delta\theta=(\theta_1-\theta_2)_{\max}=\tan^{-1}(\eta^{\frac{1}{2}}-1/\eta^{\frac{1}{2}}) \tag{6}$$

at a value of θ_2 given by

$$\tan^2\theta_2=1/\eta. \tag{7}$$

Solving (6) for η, we have

$$\eta=(1+\sin\Delta\theta)/(1-\sin\Delta\theta), \tag{8}$$

and the reflection coefficient is simply

$$\Gamma=(\eta-1)/(\eta+1)=\sin\Delta\theta. \tag{9}$$

Figure 4 is a plot of actual data taken at a wavelength of about 10 cm, with a coupling system of reflection

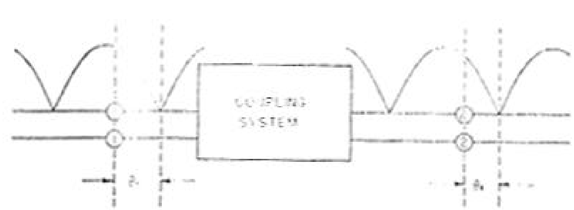

FIG. 3. Definitions of node positions θ_1 and θ_2.

[2] E. A. Guillemin, *Communication Networks* (John Wiley and Sons, Inc., New York, 1935), Vol. II.

coefficient $\Gamma=0.466$. One finds $\Delta\theta$ from the difference between maximum and minimum values of (x_1-x_2) and from this determines Γ and η. Then, as is seen from differentiating (5) at $\theta_2=0$, the positions of the input and output reference points are determined as the point where (x_1-x_2) is midway between the maximum and minimum values, with positive slope. Thus, the three parameters of the coupling system are all determined from a single plot of data.

III. THE CONCEPT OF GENERALIZED IMPEDANCE

When we come to the case of a periodically loaded wave guide (see Fig. 6 for an example), it is not immediately obvious what is meant by impedance and position of a node. In fact, the exact distinction between a running wave and a standing wave is none too clear. One thing which we need is a generalization of the concept of impedance which goes beyond the familiar generalization to wave impedance in "smooth" wave guides (i.e., wave guides in which all boundary conditions are independent of the z coordinate). The concept of impedance is useful because it gives us a certain piece of information about the configuration of electromagnetic fields, and any quantity, however defined, which conveys the same information may be called a generalized impedance.

In any guide in which only one mode type is excited, there are only two linearly independent possible fields; any type of wave, standing or running, may be expressed as a linear combination of any two linearly independent fields. This situation exists in the guides under consideration in regions far from the local fields due to coupling systems, and it is in these regions that we must be able to define what we mean by impedance or node position. Let $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$ be two different possible field configurations; then any field is specified by two complex amplitudes a_1, a_2:

$$\mathbf{E}=a_1\mathbf{E}_1+a_2\mathbf{E}_2,\qquad \mathbf{H}=a_1\mathbf{H}_1+a_2\mathbf{H}_2. \tag{10}$$

Instead of giving a_1, a_2, we might specify the product (a_1a_2) and the ratio (a_1/a_2); the former quantity is essentially a measure of the power level, while the latter specifies the field configuration to within a multiplicative constant and is therefore a generalized impedance.

The quantities a_1, a_2 are evidently very loosely analogous to voltage and current. This analogy may be greatly strengthened by a proper choice of the basis fields $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$. In the first place, we should like to have the power flow given by the usual formula

$$P=\tfrac{1}{2}Re(V\bar{I})=\tfrac{1}{2}Re(a_1\bar{a}_2), \tag{11}$$

where the bar denotes the complex conjugate. This power flow is given by an integral of the complex Poynting vector across any surface S' spanning the guide; using the expansion (10) we have

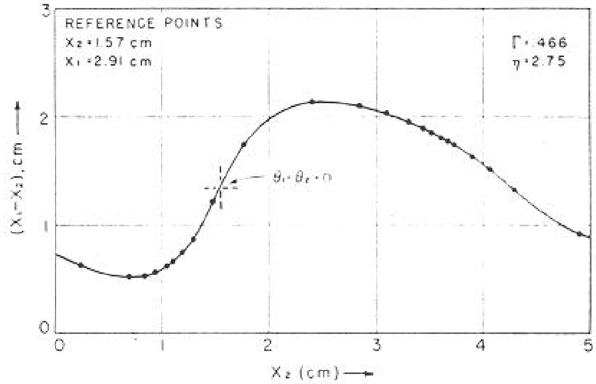

FIG. 4. Experimental plot of Eq. (5). As explained in Sec. VI, these data were actually taken with a periodically loaded guide as the output line.

$$P=\tfrac{1}{2}Re\int_{S'}(\mathbf{E}\times\bar{\mathbf{H}})\cdot d\mathbf{S}$$

$$=\tfrac{1}{2}Re\int_{S'}\left[|a_1|^2(\mathbf{E}_1\times\bar{\mathbf{H}}_1)+a_1\bar{a}_2(\mathbf{E}_1\times\bar{\mathbf{H}}_2)+\bar{a}_1a_2(\mathbf{E}_2\times\bar{\mathbf{H}}_1)+|a_2|^2(\mathbf{E}_2\times\bar{\mathbf{H}}_2)\right]\cdot d\mathbf{S}. \tag{12}$$

This will of course be independent of which particular surface S' is chosen, provided losses in the guide can be neglected. Now if the equality of (11) and (12) is to be an identity in a_1, a_2, it is evident that we must have

$$Re\int_{S'}(\mathbf{E}_1\times\bar{\mathbf{H}}_1)\cdot d\mathbf{S}=Re\int_{S'}(\mathbf{E}_2\times\bar{\mathbf{H}}_2)\cdot d\mathbf{S}=0, \tag{13}$$

whereupon (12) becomes

$$P=\tfrac{1}{2}Re\left\{a_1\bar{a}_2\int_{S'}(\mathbf{E}_1\times\bar{\mathbf{H}}_2+\bar{\mathbf{E}}_2\times\mathbf{H}_1)\cdot d\mathbf{S}\right\}. \tag{14}$$

In (14) use has been made of the fact that any term in (12) may be replaced by its complex conjugate since only the real part is taken. Equation (13) implies that we should choose basis fields that carry no power; i.e., standing waves for which $\mathbf{E}$ and $\mathbf{H}$ are everywhere $\pi/2$ out of phase. The basis fields $(\mathbf{E}_1, \mathbf{H}_1)$, $(\mathbf{E}_2, \mathbf{H}_2)$ thus correspond to two different pure reactance terminations or to two different node positions. They still contain arbitrary phase factors, however, and in order to reduce (14) to (11) we should choose them so that the integral in (14), which is in any event a constant independent of the surface S', becomes real. Proper choices of the amplitudes of the basis fields will then give us

$$\int_{S'}\left[\mathbf{E}_1\times\bar{\mathbf{H}}_2+\bar{\mathbf{E}}_2\times\mathbf{H}_1\right]\cdot d\mathbf{S}=1. \tag{15}$$

If we choose the following conditions (sufficient but

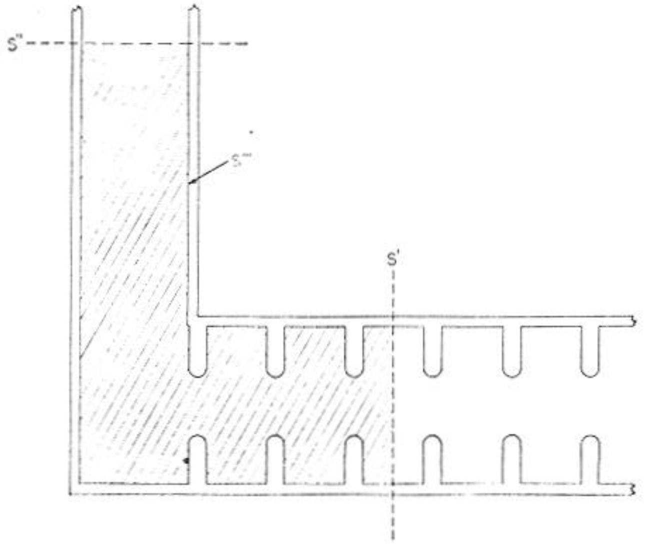

FIG. 5. Region of integration in Eq. (20). The smooth wave guide at the top is coupled to the periodically loaded one through a wide iris.

not necessary):

$$\begin{aligned}&\mathbf{E}_1,\ \mathbf{H}_2 \text{ real},\\ &\mathbf{E}_2,\ \mathbf{H}_1 \text{ imaginary},\end{aligned} \tag{16}$$

then (13) is automatically satisfied, and with proper normalization so is (15). Thus if the conditions (15), (16) are imposed on the basis fields, the power flow is given correctly by (11).

It remains to be demonstrated that a_1, a_2 are still analogous to voltage and current in the sense that the voltage and current (V, I) seen at some other part of a complicated network (in particular in a smooth transmission line on the other side of some coupling system to which the periodically loaded guide is connected) are related to a_1 and a_2 by the equations appropriate to a four-terminal network. Merely from the fact that Maxwell's equations are linear, we know that a relation of the form

$$\begin{pmatrix}V\\ I\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} \tag{17}$$

must exist; it must now be shown that the determinant of the transformation is unity:

$$\begin{vmatrix}A & B\\ C & D\end{vmatrix}=1, \tag{18}$$

and that in the case of a lossless network A and D are real while B and C are pure imaginary. The first of these follows from the reciprocity theorem; let us consider the functions $(\mathbf{E}_1, \mathbf{H}_1)$, $(\mathbf{E}_2, \mathbf{H}_2)$ as standing not only for the base functions in the periodically loaded guide, but also their analytic continuations through the coupling system and into the smooth transmission line on which V, I are measured. Then at all points inside the complete network both of these fields will satisfy Maxwell's equations

$$\nabla\times\mathbf{H}=j\omega\epsilon\mathbf{E} \quad \nabla\times\mathbf{E}=-j\omega\mu\mathbf{H}, \tag{19}$$

in which in general ϵ, μ are functions of position. If the coupling network is lossy, ϵ will be complex at some points within it. Because of (19) it is easily shown that

$$\nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1)=0$$

or

$$\oint_S(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1)\cdot d\mathbf{S}=0, \tag{20}$$

where we integrate over any closed surface. Equation (20) is an expression of the reciprocity theorem in the form given by Lorentz. Let us now carry out this integration over the surface S consisting of the surface S' spanning the periodically loaded guide, as used in Eqs. (12)–(15), the metallic walls S''' of the system extending from S' through the coupling system, and out to a plane surface S'' normal to the smooth guide on the other side of the coupling system, as shown in Fig. 5. S'' is the reference plane at which the voltage and current V, I are measured. The region enclosed by this closed surface $S=S'+S''+S'''$ is shaded in Fig. 5.

Because of the boundary conditions satisfied by $\mathbf{E}_1, \mathbf{E}_2$ on the metallic surface, the integral over S''' vanishes separately. Then, if the positive normals to S', S'' are chosen so that both point in the direction of power flow, we have

$$\int_{S''}(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1)\cdot d\mathbf{S} =\int_{S'}(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1)\cdot d\mathbf{S}.$$

But with the choice of phases given by (16) the integral over S' is the same as that in (15), which we have normalized to unity. (This incidentally gives an independent proof that the integral (15) is independent of our choice of the surface S'.) Therefore,

$$\int_{S''}(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1)\cdot d\mathbf{S}=1. \tag{21}$$

Now at the reference plane S'' the voltage and current are ordinarily defined as proportional to the transverse electric and magnetic fields, respectively. The transverse fields at S'' are therefore expressible as

$$\begin{aligned}\mathbf{E}(S'')&=V\mathbf{E}_t(S''),\\ \mathbf{H}(S'')&=I\mathbf{H}_t(S''),\end{aligned}$$

where $\mathbf{E}_t$, $\mathbf{H}_t$ are normal mode functions so normalized that power flow is given by (11). We consider the smooth transmission system to support only a single propagating mode at the frequency of operation and the reference plane S'' to be far from the local fields of the coupling system. Therefore,

$$\int_{S''}(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}=VI. \tag{22}$$

If the smooth transmission system is a coaxial line, it is easily shown that (22) is valid where V and I are the actual voltage between conductors and actual current flow, respectively. Thus we can put in general

$$\int_{S''} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{S} = V_1 I_2 - V_2 I_1 = 1, \quad (23)$$

where V_1 is the voltage at S'' when the field $\mathbf{E}_1$ is excited, I_2 is the current when $\mathbf{H}_2$ is excited, etc. But reference to (17) shows that

$$V_1 = A, \quad I_1 = C,$$
$$V_2 = B, \quad I_2 = D,$$

since, for example, A and C are the voltage and current when $a_1 = 1$, $a_2 = 0$, i.e., when the basis field $(\mathbf{E}_1, \mathbf{H}_1)$ is excited. Therefore (23) is equivalent to (18). Furthermore, it is evident that when the entire network is lossless, the choice of phases made in (16) for the region of the periodically loaded guide applies also to the analytical continuation of $\mathbf{E}_1$, $\mathbf{H}_1$, etc., to the surface S'', so that A and D are real; B and C are imaginary. Thus all of the properties of the network equations (1) which were needed for deriving the nodal shift formulas have been shown to hold for the connection (17) in which a_1, a_2 are used in place of voltage and current.

IV. CHANGE OF BASIS FIELDS

The conditions which we have imposed on the basis fields are merely that E_1, E_2 be linearly independent and that (15), (16) be satisfied. Even with these very weak restrictions, the analogy between the expansion coefficients (a_1, a_2) and the circuit quantities of voltage and current has become so complete that all of the relations of general circuit theory remain valid when (a_1, a_2) are used. The basis fields are, however, far from being uniquely determined by (15), (16); if (E_1, H_1) and (E_2, H_2) is any set allowed by our conditions, then an equally good set is (E_1', H_1'), (E_2', H_2'), where

$$\begin{aligned} E_1' &= G_{22}E_1 - G_{21}E_2, & H_1' &= G_{22}H_1 - G_{21}H_2, \\ E_2' &= -G_{12}E_1 + G_{11}E_2, & H_2' &= -G_{12}H_1 + G_{11}H_2, \end{aligned} \quad (24)$$

in which the matrix

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

has determinant unity, G_{11} and G_{22} being real while G_{12} and G_{21} are pure imaginary. These are just the conditions that the new basis fields should satisfy (15), (16) when the old ones do. A matrix with these properties will be called a G matrix. A field which has expansion coefficients (a_1, a_2) with respect to the old basis fields will have coefficients (a_1', a_2') with respect to the new ones, where

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad (25)$$

for we then find that

$$\mathbf{E} = a_1\mathbf{E}_1 + a_2\mathbf{E}_2 = a_1'\mathbf{E}_1' + a_2'\mathbf{E}_2', \text{ etc.}$$

The product of any two G matrices is another G matrix, so that the totality of all transformations of the type (25) forms an infinite group. Since the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of general circuit parameters for a lossless four-terminal network is a G matrix, we see that there is a one-to-one correspondence between all possible lossless four-terminal networks and possible changes of basis fields which preserve the analogies of (a_1, a_2) to voltage and current. Any such change of basis fields is equivalent to looking at the original voltage and current through some particular lossless network.

Since a G matrix has three independent parameters, we may expect in general that three more independent conditions may be imposed on our basis fields in addition to the above ones before they become uniquely determined. Additional conditions which would be convenient in various problems are that the two basis fields should represent the same stored energy, that the generalized impedance (a_1/a_2) should equal unity when a pure running wave is excited, etc.

V. APPLICATION TO A PERIODICALLY LOADED GUIDE

Suppose that the wave guide contains regularly spaced structures with a repetition distance "a"; in other words, the entire system is invariant under translations in the z direction through the distance a. The guide may be considered as a cascaded line of "unit cells" each of length a, but the boundaries of the cells are arbitrary. Now we can choose any two linearly independent basis fields $\mathbf{E}_1(x, y, z)$; $\mathbf{E}_2(x, y, z)$ which satisfy conditions (15) and (16). Suppressing the x, y coordinates for brevity, we write the expansion of a general field in the periodic structure as

$$\mathbf{E}(z) = a_1\mathbf{E}_1(z) + a_2\mathbf{E}_2(z). \quad (26)$$

We now inquire under what conditions $\mathbf{E}(z)$ represents a pure running wave. By this we mean that the value of $\mathbf{E}$ in one cell is a multiple of its value in an adjacent one:

$$\mathbf{E}(z+a) = e^{-j\beta}\mathbf{E}(z), \quad (27)$$

where $j\beta$ is the propagation constant per cell, which is purely imaginary in the passband. This corresponds to the condition of impedance match in the theory of iterated networks, if we consider each cell to correspond to a four-terminal network.‡ To see this, we note that the expansion (26), while valid at all values of z, may be considered as applying to one section only, say the n'th, in which case a_1, a_2 correspond to voltage and current at the output of the n'th of the iterated networks. The fields in the $(n+1)$th section may be expanded in terms of new basis fields $\mathbf{E}_1'$, $\mathbf{E}_2'$ which bear the same relation to the $(n+1)$th section as $\mathbf{E}_1$, $\mathbf{E}_2$ did to the n'th; in other words, they are obtained by translating $\mathbf{E}_1$, $\mathbf{E}_2$ one section down the pipe:

$$\mathbf{E}_1'(z)=\mathbf{E}_1(z-a),\qquad \mathbf{E}_2'(z)=\mathbf{E}_2(z-a), \tag{28}$$

and the new expansion is

$$\mathbf{E}(z)=a_1'\mathbf{E}_1'(z)+a_2'\mathbf{E}_2'(z) \tag{29}$$

so that a_1', a_2' correspond to voltage and current delivered by the $(n+1)$th network. This is a particular type of change of basis fields as discussed above, since $\mathbf{E}_1'$, $\mathbf{E}_2'$ are linearly related to $\mathbf{E}_1$, $\mathbf{E}_2$ through some G matrix:

$$\begin{pmatrix}\mathbf{E}_1\\ \mathbf{E}_2\end{pmatrix}=\begin{pmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}\begin{pmatrix}\mathbf{E}_1'\\ \mathbf{E}_2'\end{pmatrix}. \tag{30}$$

We now substitute these relations into the condition (27) for a running wave. Express $\mathbf{E}(z+a)$ by means of Eqs. (28) and (29) and $\mathbf{E}(z)$ by Eq. (26), and it assumes the form

$$\mathbf{E}(z+a)=a_1'\mathbf{E}_1(z)+a_2'\mathbf{E}_2(z)=e^{-j\beta}[a_1\mathbf{E}_1(z)+a_2\mathbf{E}_2(z)]$$

or, since $\mathbf{E}_1$, $\mathbf{E}_2$ are linearly independent,

$$a_1'=e^{-j\beta}a_1,\quad a_2'=e^{-j\beta}a_2 \tag{31}$$

$$a_1'/a_2'=a_1/a_2. \tag{32}$$

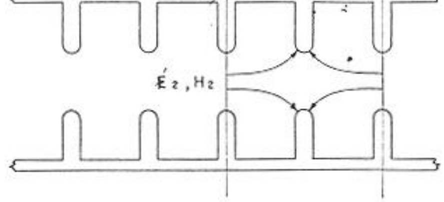

FIG. 6. Electric lines of a possible choice of basis fields for the disk-loaded linear accelerator pipe.

‡ We are, however, particularly anxious to avoid drawing equivalent circuits of the sections, because one of the principal purposes of this analysis is to show that a rigorous treatment of fields leads to relations of the same form as the ordinary circuit-theory relations independently of any approximate equivalent circuit.

The impedance is the same in each cell for a running wave, the same situation that one finds in the theory of iterated networks. As in the classical filter theory, we expect that the propagation constant $j\beta$ and the characteristic impedance (32) are determined by the frequency and the structure of the network. The only properties of the network that are needed to determine them are the elements of the G matrix in (30). This is clear from the relation (25) which, using (31), becomes

$$e^{-j\beta}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}. \tag{33}$$

Thus, $e^{-j\beta}$ is an eigenvalue, while a_1, a_2 are the components of the corresponding eigenvector, of the G matrix connecting the original basis fields with their translations through one period. By standard methods, we obtain the explicit solutions

$$e^{-j\beta}=\tfrac{1}{2}\{(G_{11}+G_{22})\pm[(G_{11}+G_{22})^2-4]^{\frac{1}{2}}\}, \tag{34}$$

$$a_1/a_2=(e^{-j\beta}-G_{22})/G_{21}. \tag{35}$$

The two values of β from (34) satisfy the relation $\beta'=-\beta''$, corresponding to the two directions of propagation. In the passbands β is real, and we have

$$\cos\beta=\tfrac{1}{2}(G_{11}+G_{22})<1. \tag{36}$$

We now observe that relations (34), (35) may be greatly simplified by a particular choice of basis fields. We may choose $\mathbf{E}_1$ and $\mathbf{E}_2$ to differ essentially only by a translation of one period down the guide, provided that the fields so defined are linearly independent. (In general, this will be the case, but the edges of the passbands are critical frequencies at which $\mathbf{E}_1(z)=\pm\mathbf{E}_1(z+a)$ so that linear independence fails.) In addition, we must introduce a 90° phase shift as required by condition (16). Therefore, let us define E_2 by

$$\mathbf{E}_2(z)=-j\mathbf{E}_1(z-a)=-j\mathbf{E}_1'(z). \tag{37}$$

A possible choice of these fields for the Stanford linear accelerator is sketched in Fig. 6. Then the G matrix in (30) becomes

$$G=\begin{pmatrix}G_{11} & -j\\ -j & 0\end{pmatrix}, \tag{38}$$

while (34), (35), and (36) reduce to

$$e^{-j\beta}=G_{11}/2\pm[(G_{11}/2)^2-1]^{\frac{1}{2}}, \tag{39}$$

$$(a_1/a_2)=je^{-j\beta}, \tag{40}$$

$$\cos\beta=G_{11}/2. \tag{41}$$

The characteristic impedance and propagation constant are not essentially different quantities with this choice of basis fields. At the edges of the passband, $\beta=0$ or π, and the characteristic impedance becomes purely imaginary; while at the center of the passband, $\beta=\pi/2$, and the characteristic impedance is unity.

To sum up the results found above, there is a complete analogy between wave propagation in a periodic structure and the classical theory of filters composed of iterated four-terminal networks. The device of expanding fields in terms of a set of linearly independent basis fields has enabled us to find coefficients a_1, a_2 which satisfy the same relations as do voltage and current in lumped-constant network theory. This is not just a coincidence, for ordinary circuit theory is a special case of the analysis presented here. In any lumped-constant four-terminal network we can define fields $(\mathbf{E}_1, \mathbf{H}_1)$ to be those resulting from unit voltage, zero current at the output (i.e., open-circuited load), while $(\mathbf{E}_2, \mathbf{H}_2)$ are the fields resulting from unit current, zero voltage at the same terminals. Then any possible state of excitation of the network is described by the expansion (26) in which a_1, a_2 are the output voltage and current in the ordinary sense. The G matrix giving the rule of translation of basis fields through one section then reduces to the matrix of general circuit parameters A, B, C, D as treated by Guillemin.[2] Thus, by introducing the concept of basis fields and then allowing ourselves a greater freedom of choice in their definitions than is usual (the extent of this freedom being precisely described by the group of G matrices), we create a rather sweeping generalization of ordinary circuit-theory, which applies without approximation to any periodic structure.

VI. EXPERIMENTAL MEASUREMENTS

We are now able to say just what it is that we want a coupling system to the periodically loaded guide to do; when load conditions are so adjusted that we have a pure running wave in it, as in the Stanford linear accelerator, we want a match to appear also in the smooth transmission line on the other side of the coupling system. The problem is to find an experimental procedure, using only pure reactance terminations of the loaded wave guide, that will tell us what the input impedance on the smooth transmission line would be if there were a running wave in the loaded guide.

We may produce all reactive values of the impedance (a_1/a_2) in the loaded guide by sliding a metal shorting-plug into it at various distances, as shown in Fig. 7. However, we do not at the outset know how this reactance is related to the position of the shorting plug. This may be found from the behavior of the impedance seen on the other side of the coupling system as follows. First we define the input reference plane and the basis fields in a manner analogous to the definitions of Sec. II:

(a) Match the periodic guide: $a_1/a_2 = j\exp(-j\beta)$, and choose the input reference plane at a point on the input line where the impedance seen looking toward the coupling system is $V/I = Kj\exp(-j\beta)$, with K real. If β is very small (i.e., the repetition distance a is very small compared to a wavelength), such a point may not exist, in which case we redefine the cell to contain n of the small cells, $n = 2, 3, 4, \cdots$. This

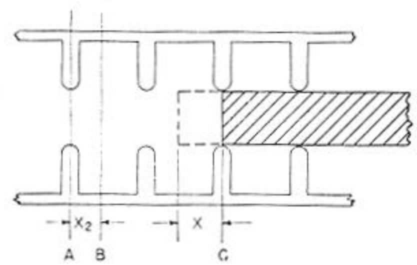

FIG. 7. Metallic plunger for varying load reactance, with definitions of quantities used in Fig. 9. The actual plunger is a brass tube with fingers that ensure good metallic contact when a disk is reached.

multiplies β by n, and a suitable value of n can always be found such that this reference plane will exist.

(b) Find the plunger position which places a voltage node at this input reference plane, and define the fields in the periodic guide for this plunger position (and sufficiently far from the local fields of the coupling system and plunger) as $(\mathbf{E}_2, \mathbf{H}_2)$.

Then we have, remembering that A and D are real and B and C are imaginary,

$$\text{from (b)}: \quad B = 0,$$
$$\text{from (a)}: \quad C = 0, \quad K = A/D;$$

and the impedance relation, for this choice of reference plane and basis fields, again reduces to a simple proportionality

$$V/I = K(a_1/a_2). \tag{42}$$

By transmission-line theory, the SWR of the coupling system is then

$$\eta = \tfrac{1}{2}(K + K^{-1})\csc\beta + \left[\tfrac{1}{4}(K + K^{-1})^2\csc^2\beta - 1\right]^{\frac{1}{2}}, \tag{43}$$

which, for $\beta = \pi/2$, reduces to $\eta = K$.

As in Sec. II, the above are *definitions* of reference plane and basis fields, not experimental methods for finding them, because by hypothesis we have no experimental way of knowing when the periodic guide is matched until the parameters of some coupling system have been found. We can find these parameters as follows: First, note that when the plunger is moved in through one repetition period (as defined in (a) above), the basis fields $\mathbf{E}_1, \mathbf{H}_1$ are then excited so that there is an infinite impedance at the input reference plane; experimentally, pushing the plunger in through the distance a moves the node in the smooth transmission line just a quarter-wavelength. Suppose we measure the node position x_1 on the input line as a function of the plunger position x and plot a graph of $x_1(x) - x_1(x+a)$ *versus* x. At the point where this crosses $(\lambda/4)$, x_1 is the position of the input reference plane, x is the plunger position which creates the basis fields $(\mathbf{E}_2, \mathbf{H}_2)$, and $(x+a)$ is the plunger position which creates $(\mathbf{E}_1, \mathbf{H}_1)$. (Note that according to Fig. 7, the direction of increasing x is opposite to the direction of increasing z as used in Sec. V. The reason for this is experimental convenience.)

The analogy between the relations of this section and those of Sec. II may be strengthened if we define an effective node position θ_2 in the periodic guide for any plunger position by

$$j\tan\theta_2 = (a_1/a_2).$$

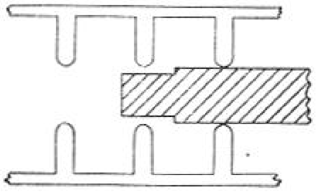

FIG. 8. Modification of plunger to avoid contact discontinuities.

Since, as before, the node position on the input line is given by $j\tan\theta_1=V/I$, Eq. (42) assumes a form analogous to (4):

$$\tan\theta_1=K\tan\theta_2. \tag{44}$$

The remaining quantities which we wish to find are the effective node position θ_2 (or what is more convenient experimentally, the length $x_2=a\theta_2/\beta$) as a function of plunger position x, the impedance magnification constant K, and the phase-shift per section, β. To find these, let us express the fields as functions of the longitudinal coordinate z, and the plunger position x:

$$\mathbf{E}(z,x)=a_1(x)\mathbf{E}_1(z)+a_2(x)\mathbf{E}_2(z).$$

If we pull the plunger out by a distance a_1, the new fields are just translated in the $+z$ direction by an amount a:

$$\mathbf{E}(z,x-a)=a_1(x-a)\mathbf{E}_1(z)+a_2(x-a)\mathbf{E}_2(z)$$
$$=\mathbf{E}(z-a,x)=a_1(x)\mathbf{E}_1(z-a)+a_2(x)\mathbf{E}\_(z-a).$$

From this, using (30), we obtain

$$\begin{pmatrix}a_1(x-a)\\a_2(x-a)\end{pmatrix}=\begin{pmatrix}0 & j\\ j & G_{11}\end{pmatrix}\begin{pmatrix}a_1(x)\\a_2(x)\end{pmatrix}. \tag{45}$$

Now calling the plunger position $x=0$ at the position which generates the basis fields $(\mathbf{E}_2, \mathbf{H}_2)$, we find that

$(a_1/a_2)_{x=0}=0$ (definition),

$(a_1/a_2)_{x=a}=\infty$ (in agreement with the previous discussion),

$$z\ (a_1/a_2)_{x=-a}=j/G_{11}, \tag{46}$$

$$(a_1/a_2)_{x=-2a}=jG_{11}/(G_{11}^2-1). \tag{47}$$

Taking the product and ratio of (46), (47) we have, using the relation $j\tan\theta_1=V/I$ and (42),

$$\frac{(\tan\theta_1)_{x=-a}}{(\tan\theta_1)_{x=-2a}}=1-\frac{1}{G_{11}^2}, \tag{48}$$

$$(\tan\theta_1)_{x=-a}\cdot(\tan\theta_1)_{x=-2a}=K^2/(1-G_{11}^2), \tag{49}$$

from which $G_{11}=2\cos\beta$ and K may be determined. Once K is known, we can use Eq. (42) backwards to find the effective node position $x_2(x)$ from the data on x_1 *versus* x already taken. Thus, once more, from an experimental determination of x_1 as a function of x, we can deduce the SWR of the coupling system [using Eq. (43)], the propagation constant of the periodic guide, the position of the reference planes at which the impedance relations are particularly simple, and the effective node position in the periodic guide as a function of plunger position. Most of the work at Stanford has been done at that frequency for which $\beta=\pi/2$, where the situation is greatly simplified, as is seen from inspection of the above equations for that case.

The use of a metallic plunger with a flat end as in Fig. 7 was found to have a disadvantage that when metallic contact is made to a disk in pushing it forward, there is a small but annoying discontinuity in the function $x_1(x)$. However, one can shape the plunger as in Fig. 8, and it is possible to find a length for the extension (1.3 cm in our case) at which no measurable discontinuity in $x_1(x)$ is found. The condition for this is, of course, that there should be no rf voltage across the gap just before it closes. A theoretical reason why the extension should produce this condition has not been sought; we merely record it as an experimental fact that it is not at all difficult to find the proper length.

The function $x_2(x)$ is a universal one, depending on the periodic guide, the frequency, and the plunger, but not on the particular coupling system with which it was obtained. Therefore, once one has this universal curve (Fig. 9 applies to the present linear accelerator

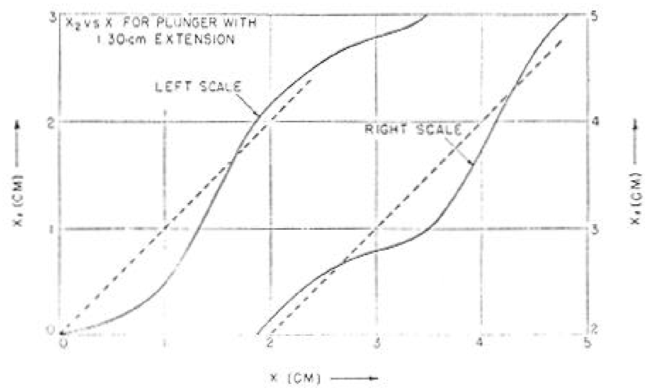

FIG. 9. Effective node position as a function of plunger position for the Stanford linear accelerator guides.

guide in use at Stanford) one can return to the method of Sec. II and obtain the parameters of a number of different coupling systems by plotting (x_1-x_2) *versus* x_2. The data of Fig. 4 were taken in this way.

One tricky point which nearly everyone finds disconcerting at first sight concerns the direction of motion of the effective node position. According to Fig. 9 and the definitions in Fig. 7, moving the plunger to the *left* moves the effective node to the *right*, whereas one naïvely expects the opposite behavior. The direction of node motion depends on the details of the fields around the plunger, and one can show that the general criterion is as follows: Push the plunger in through an infinitesimal distance and consider the fields in the small volume that was removed in front of the plunger. If before the movement there was more energy stored in electric than in magnetic fields in this volume, then the effective node moves in the opposite direction to the plunger, and vice versa.

The writer is indebted to the late Dr. W. W. Hansen, under whose direction the experimental part of this work was done, for many helpful discussions.