Kramers–Kronig relationship between ultrasonic attenuation and phase velocity

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Kramers–Kronig relations linking the attenuation and dispersion are presented for a linear acoustic system. These expressions are used as a starting point to derive approximate, nearly local expressions relating the ultrasonic attenuation at a specific frequency to the local frequency derivative of the phase velocity (i.e., dispersion). The validity of these approximate relationships is demonstrated in several acoustic systems exhibiting substantially different physical properties.

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INTRODUCTION

Quantitative relationships between attenuation and the frequency dependence of the phase velocity (dispersion), the validity of which depend only upon the properties of linearity and causality of the system under investigation, have proven useful in a number of settings. Examples include the Kramers–Kronig relationships[1-4] connecting the in-phase and out-of-phase components of the appropriate susceptibility in electromagnetic and acoustic phenomena and the Bode relationship[5] connecting the gain and phase shift in amplifier circuits. In a previous short report we discussed quantitative relationships between the attenuation and dispersion for a linear acoustic system.[6] The present study presents a derivation of the generalized relationship between the ultrasonic attenuation and dispersion based on linear response theory. In addition, useful approximate forms for these relations are derived under the conditions that the attenuation and the dispersion do not vary rapidly as functions of frequency. The use of these approximate relationships for a wide range of acoustic systems is illustrated and the implications of the existence of these relations concerning investigation of the physical mechanisms responsible for the observed attenuation are examined.

In Sec. I formulas are derived for predicting the ultrasonic attenuation as a function of frequency from measurements of the frequency dependence of the dispersion and vice versa. In Sec. II we illustrate the use of the approximate relations, including a theoretical analysis of a simple relaxation. In this section we also compare the results of experiments with predictions based on the theory of Sec. I in a number of different acoustic systems. Section III is devoted to a discussion of the implications of the existence of these generalized relationships as regards the study of mechanisms describing the propagation of ultrasound in materials. Specifically, the formulas developed can be used to predict ultrasonic properties not yet measured and to set limits on properties which may lie beyond the range of currently available measurement techniques. In addition, the existence of these completely general relationships renders invalid attempts to compare the attenuation and dispersion as a means of validating some specific model proposed to explain the attenuation. As noted in the previous short report, a demonstration of agreement between the dispersion and attenuation predicted from a specific model and the measured dispersion and attenuation serves only to establish that the specific mechanism and the system as measured satisfy the conditions of causality (i.e., that effects do not precede their causes) and linearity (i.e., that to some approximation the response of the system is linearly proportional to the stimulus).[6]

I. THEORY

We consider an isotropic acoustic medium described by a generalized form of Hooke's law, and analyze the response of this medium to an incident acoustic wave in terms of linear response theory. Utilizing the formalism of linear response theory, we review the derivation of the Kramers–Kronig equations which relate the in-phase and out-of-phase components of the response of the acoustic system. We then use this result in conjunction with the dispersion relation for the propagation of sound in the medium to obtain general expressions which relate the frequency dependence of the attenuation coefficient to the frequency dependence of the phase velocity. Finally, we examine these expressions for the case in which the phase velocity and the attenuation coefficient do not vary rapidly with frequency. Under these conditions, the general relations are cast into a more useful expression which relates the attenuation coefficient to the dispersion over a limited frequency range.

In the Hooke's law limit, the ultrasonic equation of motion describes a linear system whose response can be represented by

$$s(t) = \int_{-\infty}^{\infty} K(t-t')p(t')\,dt', \tag{1}$$

where s is the condensation, p is the pressure, and K is the adiabatic compressibility. In Eq. (1), $K(t-t')$ plays the role of generalized susceptibility relating the response $s(t)$ to the stimulus $p(t')$. The central theme of this paper is that the principle of causality places restrictions on the behavior of $K(t-t')$. Specifically, the response $s(t)$ can depend upon past but not upon future values of the stimulus $p(t')$. The restrictions imposed by causality on the generalized susceptibility are conveniently expressed in terms of the real and imaginary parts of the Fourier transform of the compressi-

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bility. If the Fourier transform exists for s, K, and p, where the Fourier transform for $K(t)$ is defined as

$$K(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}K(\omega)e^{-i\omega t}, \tag{2}$$

then the response of the system in the frequency domain becomes

$$S(\omega)=K(\omega)P(\omega), \tag{3}$$

where $K(\omega)$ is the frequency domain compressibility. In general, $K(\omega)$ is a complex function, $K(\omega)=K_1(\omega)+iK_2(\omega)$, where $K_1(\omega)$ is the real part and $K_2(\omega)$ is the imaginary part of the compressibility.

To determine the relationship between the ultrasonic attenuation coefficient and the dispersion, we begin by investigating the frequency domain response of an acoustic system to a pressure fluctuation applied as an impulse, i.e., a delta function. Under the action of this delta function, the condensation becomes

$$s(t)=K(t)=\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{-i\omega' t}K(\omega'). \tag{4}$$

The Kramers–Kronig relations can be obtained from Eq. (4) by applying two conditions. First, the compressibility, $K(t)$, is physically measurable and thus must be a real function. For this condition to be satisfied, $K(-\omega)$ must equal $K^*(+\omega)$. That is, the real part of the transform of the compressibility must be a symmetric function and the imaginary part an antisymmetric function of frequency. Applying this constraint to Eq. (4), the condensation reduces to

$$s(t)=\frac{1}{\pi}\left(\int_0^{\infty}d\omega' K_1(\omega')\cos\omega' t+\int_0^{\infty}d\omega' K_2(\omega')\sin\omega' t\right). \tag{5}$$

The second requirement on $K(t)$ is that of causality, which states that the present condensation should not depend on the future pressure. Consequently, $s(t)$ is zero for $t<0$, which from Eq. (5) requires that

$$\int_0^{\infty}d\omega' K_1(\omega')\cos\omega' t+\int_0^{\infty}d\omega' K_2(\omega')\sin\omega' t=0,\quad t<0, \tag{6}$$

or for $t>0$

$$\int_0^{\infty}d\omega' K_1(\omega')\cos\omega' t-\int_0^{\infty}d\omega' K_2(\omega')\sin\omega' t=0,\quad t>0. \tag{7}$$

To cast this result into a more useful form, we multiply (7) by $\exp(-\lambda t)$, where λ equals $\epsilon+i\omega$, and integrate over positive times to obtain

$$\int_0^{\infty}\frac{\lambda K_1(\omega')-\omega' K_2(\omega')}{\omega'^2+\lambda^2}d\omega'=0,\quad \epsilon>0. \tag{8}$$

We now consider the limit as ϵ approaches 0, that is as λ approaches $i\omega$. Focusing on the denominator of Eq. (8) we observe that

$$(\omega'^2+\lambda^2)^{-1}\to P(\omega'^2-\omega^2)^{-1}-i\pi\delta(\omega'^2-\omega^2)$$
$$=P(\omega'^2-\omega^2)^{-1}-\frac{i\pi}{2\omega}\left[\delta(\omega'-\omega)+\delta(\omega'+\omega)\right], \tag{9}$$

where P stands for the principal part in a subsequent integration.[7,8] Using Eq. (9), the real and imaginary parts of (8) then go into

$$K_1(\omega)=\frac{2}{\pi}P\int_0^{\infty}\frac{\omega' K_2(\omega')}{\omega'^2-\omega^2}d\omega', \tag{10}$$

$$K_2(\omega)=-\frac{2}{\pi}P\int_0^{\infty}\frac{\omega K_1(\omega')}{\omega'^2-\omega^2}d\omega'. \tag{11}$$

Equations (10) and (11) are the Kramers–Kronig relations for longitudinal waves. In a real acoustic system, inertia will ensure that the response and hence the generalized susceptibility to which the response is proportional falls off rapidly enough at high frequencies so that the integrals converge. In the original Kramers–Kronig application (electric susceptibility), allowance was also made for an instantaneous component of response $K_\infty\delta(t)$, where K_∞ is the value of the generalized susceptibility at arbitrarily high ("infinite") frequency. The causality condition was applied to the remainder of the total response $[K(t)-K_\infty\delta(t)]$. A similar approximation is useful in the acoustic problem, in which case $K_1(\omega')$ in the above is to be replaced by $[K_1(\omega')-K_\infty]$.

These equations can be used to relate the attenuation coefficient and phase velocity because the frequency dependent compressibility also must satisfy the dispersion relation for acoustic wave propagation

$$k^2=\omega^2\rho_0 K(\omega), \tag{12}$$

where k is the wavenumber of the ultrasonic wave and ρ_0 is the density of the medium. To satisfy Eq. (12), k must be complex. We make the identification that the wavenumber k equals $\omega/C(\omega)+i\alpha(\omega)$, where $C(\omega)$ is the phase velocity and $\alpha(\omega)$ is the attenuation coefficient. Thus, for example, a plane wave traveling in the $+x$ direction propagates as $e^{i(kx-\omega t)}=e^{-\alpha x}e^{i[\omega x/C(\omega)-\omega t]}$. The compressibility can be related to the attenuation coefficient and phase velocity such that

$$\frac{\omega^2}{C^2(\omega)}-\alpha^2(\omega)+\frac{2i\omega\alpha(\omega)}{C(\omega)}=\omega^2\rho_0[K_1(\omega)+iK_2(\omega)], \tag{13}$$

or

$$\frac{\omega^2}{C^2(\omega)}-\alpha^2(\omega)=\omega^2\rho_0 K_1(\omega), \tag{14a}$$

$$\frac{2\alpha(\omega)}{C(\omega)}=\omega\rho_0 K_2(\omega). \tag{14b}$$

In the usual case in which the magnitude of the imaginary part of the wavenumber is much less than the magnitude of the real part [i.e., $\alpha(\omega)C(\omega)/\omega\ll 1$] for all frequencies, the set of equations [Eq. (14)] decouples and the phase velocity and attenuation coefficient are determined by

$$C(\omega)=1/[\rho_0 K_1(\omega)]^{1/2}, \tag{15a}$$

$$\alpha(\omega)=[\rho_0 C(\omega)/2]\omega K_2(\omega). \tag{15b}$$

Equations (10), (11), and (15) represent a complete description of the frequency domain response of the acoustic system in terms of the phase velocity and the attenuation coefficient.[9] Using these equations, the phase velocity can be computed exactly at all frequencies if the attenuation coefficient is known at all fre-

quencies. Similarly, a knowledge of the phase velocity at all frequencies can be used to compute the attenuation at all frequencies with Eqs. (10), (11), and (15). The full Kramers–Kronig relations have proved useful in a number of settings where the dominant contribution to the infinite integrals is highly localized in frequency. For example, the exact relations are widely applied in the field of magnetic resonance, where the resonance constitutes the dominant contribution to the integral. However, these expressions appear to be of limited usefulness in settings where the integrals are not highly localized since the computation of one variable appears to necessitate a knowledge of the complementary variable for all frequencies; i.e., Eqs. (10) and (11) are nonlocal.

We now utilize the analogy between the acoustic Kramers–Kronig relation and the relationship between the frequency dependence of the gain and phase shift of an electrical amplifier to obtain a more useful expression relating the frequency dependence of the attenuation to the dispersion over a limited frequency range in nonresonant systems. Starting with the Kramers–Kronig relation connecting the gain and phase shift of an electrical amplifier, Bode demonstrated that at any frequency the phase shift is approximately related to the local rate of change of the gain with frequency.[5] The approximation is quite accurate if both the gain and phase shift are sufficiently well behaved (e.g., exhibit no resonances) over a limited frequency range centered at the frequency of interest. A similar relation between the attenuation coefficient and the phase velocity can be derived starting with Eq. (11) and using a change of variable to evaluate the integral. We define the variable $x = \ln(\omega'/\omega)$ and consider the integral of Eq. (11). The imaginary part of the compressibility becomes

$$K_2(\omega) = -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{G(x) - G(\infty)}{e^{x} - e^{-x}}\, dx\,, \tag{16}$$

where $G(x) = K_1(\omega')$ and $K_1(\infty) = G(\infty)$ since x is infinite for ω' equal to infinity. Integrating Eq. (16) by parts, we find that $K_2(\omega)$ reduces to

$$K_2(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dG(x)}{dx} \ln\coth\left(\frac{|x|}{2}\right) dx\,. \tag{17}$$

The integral in Eq. (17) can be cast into an approximate local form due to the character of the function $\ln\coth(|x|/2)$, which is illustrated in Fig. 1. As is evident from this figure, the function has a sharp singularity at $x = 0$, and thus the magnitude of the integral in Eq. (17) is dominated by the value of the integrand at $x = 0$. Consequently, if the integral is rewritten in the form

$$K_2(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} F(x) \ln\coth\left(\frac{|x|}{2}\right) dx\,, \tag{18}$$

where $F(x)$ equals $dG(x)/dx$, then $F(x)$ can be expanded about $x = 0$ to find an approximation to the integral. Expanding $F(x)$ about $x = 0$, and noting that the integral over odd powers of x in the expansion vanishes since $\ln\coth(|x|/2)$ is an even function of x, Eq. (18) becomes

$$K_2(\omega) = -\frac{2}{\pi}\sum_{n=0}^{\infty} \frac{F^{2n}(0)}{(2n)!}\int_{0}^{\infty} x^{2n} \ln\coth\frac{|x|}{2}\, dx\,. \tag{19}$$

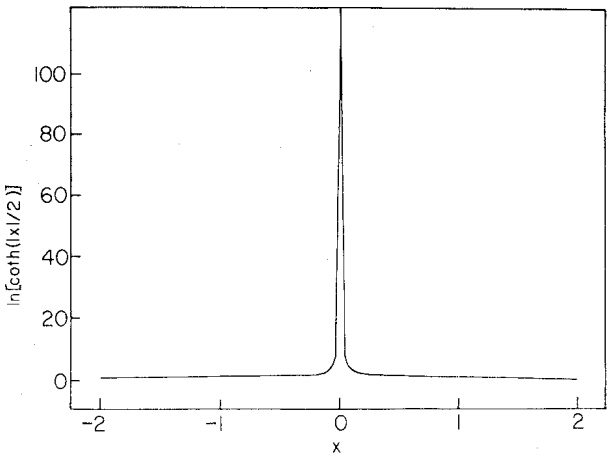

FIG. 1. This figure illustrates the character of the function $\ln[\coth(|x|/2)]$ in the neighborhood of x equals zero.

In Eq. (19) the term $F^{2n}(0)$ corresponds to the $2n$th derivative of $F(x)$ evaluated at $x = 0$. Expanding $\ln\coth(|x|/2)$ in powers of e^{-x}, the integral of Eq. (19) can be evaluated:

$$K_2(\omega) = -\frac{4}{\pi}\left[\sum_{n=0}^{\infty} F^{2n}(0)\left(\sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n+2}}\right)\right]. \tag{20}$$

Equation (20) indicates that $K_2(\omega)$ is related to the sum of the even derivatives of $F(x)$ evaluated at $x = 0$. If both the phase velocity and attenuation coefficient are slowly varying functions of frequency, then this sum can be approximated by the first few terms. Under these conditions,

$$K_2(\omega) = -\frac{4}{\pi}\left(\frac{\pi^2}{8} F(0) + \frac{\pi^4}{96} F''(0) + \cdots\right), \tag{21}$$

or substituting for $F(x)$,

$$K_2(\omega) = -\frac{\pi}{2}\left.\frac{dG(x)}{dx}\right|_{x=0} - \frac{\pi^3}{24}\left.\frac{d^3G(x)}{dx^3}\right|_{x=0} + \cdots\,. \tag{22}$$

As is demonstrated below, $dG(x)/dx$ is related to the dispersion, $dC(\omega)/d\omega$. Correspondingly, all higher derivatives of $G(x)$ are related to higher derivatives of the phase velocity with respect to frequency. Consequently, if the change in dispersion is small over a limited frequency range (e.g., no sharp resonances are present over the frequency range of interest) then the higher order derivatives can be neglected in the expansion presented in Eq. (22). The leading term in Eq. (22) can be rewritten as

$$\left.\frac{dG}{dx}\right|_{x=0} = \frac{dK_1(\omega)}{d\omega}\left.\frac{d\omega}{dx}\right|_{x=0} = \omega\frac{dK_1(\omega)}{d\omega}\,, \tag{23}$$

and $K_2(\omega)$ becomes

$$K_2(\omega) = -\frac{\pi}{2}\,\omega\frac{dK_1(\omega)}{d\omega}\,. \tag{24}$$

Equation (24) relates the imaginary part of the compressibility at a frequency ω to the local rate of change of the real part of the compressibility at the same frequency. Using Eq. (15) to relate the real part of the compressibility to the phase velocity, the derivative of

$K_1(\omega)$ with respect to frequency becomes

$$\frac{dK_1(\omega)}{d\omega} = -\frac{2}{\rho_0 C^3(\omega)} \frac{dC(\omega)}{d\omega} . \tag{25}$$

The dispersion and the attenuation coefficient can now be related by combining Eqs. (15b), (24), and (25) with the result that $dC(\omega)/d\omega$ becomes

$$\frac{dC(\omega)}{d\omega} = 2C^2(\omega)\alpha(\omega)/\pi\omega^2 , \tag{26}$$

and $\alpha(\omega)$ becomes

$$\alpha(\omega) = \frac{\pi\omega^2}{2C^2(\omega)} \frac{dC(\omega)}{d\omega} . \tag{27}$$

If we rewrite Eq. (26) as

$$\frac{dC(\omega)}{C^2(\omega)} = \frac{2\alpha(\omega)}{\omega^2} d\omega , \tag{28}$$

and integrate both sides from some reference frequency ω_0 to ω, then the phase velocity is related to the attenuation coefficient according to

$$\frac{1}{C_0} - \frac{1}{C(\omega)} = \frac{2}{\pi} \int_{\omega_0}^{\omega} \frac{\alpha(\omega')}{\omega'^2} d\omega' , \tag{29}$$

where C_0 is the sound velocity at ω_0. Equations (27) and (29) represent nearly local generalized ultrasonic attenuation-dispersion relations. The magnitude of the dispersion is usually small, so that these expressions can be further simplified to

$$\alpha(\omega) = \frac{\pi\omega^2}{2C_0^2} \frac{dC(\omega)}{d\omega} , \tag{30a}$$

$$\Delta C = C(\omega) - C_0 = \frac{2C_0^2}{\pi} \int_{\omega_0}^{\omega} \frac{\alpha(\omega')}{\omega'^2} d\omega' , \tag{30b}$$

where $C(\omega)$ is written as $C_0 + \Delta C(\omega)$ with $\Delta C(\omega) \ll C_0$, and only terms of order $\Delta C(\omega)$ are retained. In the next section, the validity of Eq. (30) in several different acoustic systems is discussed.

II. VERIFICATION OF THE NEARLY LOCAL RELATIONSHIPS

In the previous section we derived approximate forms for the Kramers–Kronig relations linking the attenuation coefficient at a frequency ω to the local rate of change of the phase velocity with frequency. These nearly local forms should represent an accurate description of the relationship between the attenuation and the dispersion in the absence of rapid variations with frequency such as those associated with a sharp resonance. In this section we explore the validity of these expressions in several physical systems.

Relaxational phenomena represent an important class of loss mechanisms. Although arising from many different physical sources, the class of loss mechanisms associated with relaxation results in frequency dependent attenuation of the form

$$\alpha(\omega)/\omega = R_0[(\omega/\omega_0)/(1+\omega^2/\omega_0^2)] . \tag{31}$$

In Eq. (31), $\alpha(\omega)/\omega$ is the attenuation per cycle, R_0 is a frequency independent constant, and ω_0 is the relaxation frequency. The attenuation per cycle for a single relaxation normalized to its value at the relaxation frequency is illustrated as a function of frequency in Fig. 2(a). Using the form of the attenuation per cycle [Eq. (31)] illustrated in Fig. 2(a), we have computed the dispersion according to the exact, nonlocal Kramers–Kronig relationship [Eqs. (10) and (15)] and according to the approximate, nearly local relationship [Eq. (29)]. In Fig. 2(b) we compare the dispersion computed from the nearly local approximate form to the dispersion obtained from the exact Kramers–Kronig relation. This figure clearly indicates that both the character and the numerical magnitude of the dispersion associated with a single relaxation is accurately described by the approximate nearly local relation derived in the previous section.

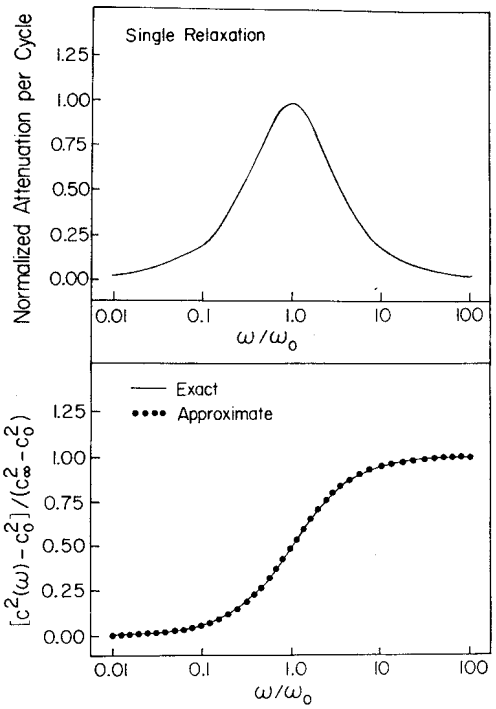

FIG. 2. The attenuation per cycle for a single relaxation normalized to its value at the relaxation frequency is illustrated as a function of frequency in the top panel. In the lower panel, the dispersion obtained from the approximate, nearly local relation [Eq. (29)] is compared to the dispersion obtained from the exact Kramers–Kronig relation.

To further test the validity of the approximate relations in describing the relationship between attenuation and dispersion in materials exhibiting relaxation, we investigate the properties of a system consisting of a solution of $CoSO_4$ in water. The attenuation coefficient and dispersion were measured in this system over the range of approximately 1 to 10 MHz by Carstensen.[10] In the top panel of Fig. 3 we present the product of the attenuation coefficient times the wavelength measured in a 1 molar solution of $CoSO_4$ in water over the frequency range of 500 kHz to 10 MHz. For small values of the dispersion, the attenuation per cycle is simply proportional to $\alpha\lambda$. From the top panel of Fig. 3 it appears that for frequencies less than 3 MHz, the attenuation per cycle is accurately described by a simple relaxation, whereas above 5 MHz, the data can be fit

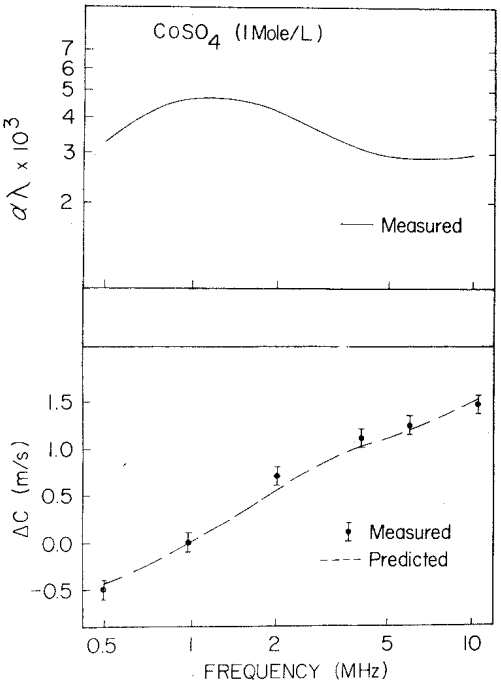

FIG. 3. The attenuation times the wavelength measured in a one molar $CoSO_4$ solution by Carstensen is illustrated in the top panel. In the lower panel, the dispersion predicted by applying Eq. (30) to the attenuation data of the top panel is compared to the measured dispersion.

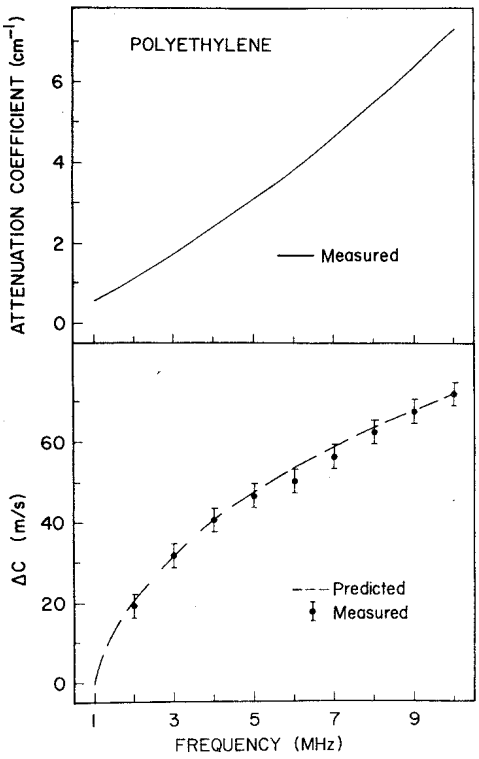

FIG. 4. The top panel shows the attenuation measured in a polyethylene plate. In the lower panel, the dispersion predicted by applying Eq. (30) to the attenuation data of the top panel is compared to the measured dispersion.

to a straight line [i.e., $\alpha(\omega)$ proportional to ω^2]. In the lower panel of Fig. 3 we compare the change in the sound velocity from its value at 1 MHz as measured by Carstensen to that predicted by Eq. (30) using the attenuation data of the top panel. Both the qualitative character and numerical value of the dispersion measured in solutions of $CoSO_4$ in water are accurately predicted by the approximate relations derived in the previous section.

We also investigated the validity of the approximate forms of the Kramers–Kronig relations by comparing the attenuation and dispersion measured from 1 to 10 MHz in polyethylene. We chose polyethylene for this investigation because it represents a system that exhibits acoustic properties very different from those of the systems described above. Specifically, polyethylene is a solid and exhibits losses which are relatively large in magnitude over the low MHz range ($\alpha\lambda$ equals 0.13 at 5 MHz). We measured the attenuation coefficient and the velocity of sound using standard techniques.[11,12] Errors in the attenuation coefficient determination were estimated to be less than 5% and errors in the velocity determination less than 1 part in 10^4. The attenuation coefficient measured in a polyethylene plate is plotted as a function of frequency over the range 1 to 10 MHz in the top panel of Fig. 4. As illustrated in this figure, the attenuation coefficient is nearly a linear function of frequency over this frequency range. Consequently, according to Eq. (30), ΔC should be a nearly logarithmic function of frequency over the same frequency range. In the bottom panel of Fig. 4, ΔC as determined experimentally is compared with that obtained from Eq. (30) (dashed curve) using the attenuation data of the upper panel. The frequency corresponding to C_0 was taken to be 1 MHz. Not only is ΔC a nearly logarithmic function of frequency in polyethylene, as predicted, but over a decade in frequency the numerical magnitude of ΔC predicted by Eq. (30) is nearly identical to that measured. These results clearly indicate that the approximate nearly local relations are also valid in polyethylene over the range 1 to 10 MHz.

III. DISCUSSION

In this study we considered acoustic propagation in a system which satisfies Hooke's law in the context of linear response theory, where the compressibility is identified as the generalized susceptibility of the linear acoustic system. Because an acoustic medium represents a causal system, the real and imaginary parts of the frequency dependent compressibility are related by the Kramers–Kronig relations. Using the dispersion relation for acoustic propagation, we have shown that the real part of the compressibility is related to the phase velocity and the imaginary part of the compressibility is related to the attenuation coefficient. Consequently, the Kramers–Kronig relations can be used in conjunction with the dispersion relation to obtain expressions linking the phase velocity to the attenuation coefficient. These nonlocal expressions are exact, and are independent of the particular mechanism responsible for the attenuation. In Sec. I we demonstrated that the exact, nonlocal Kramers–Kronig relations could be approximated by nearly local relations linking the attenuation and the dispersion in systems which do not exhibit rapid frequency variations. The validity of the

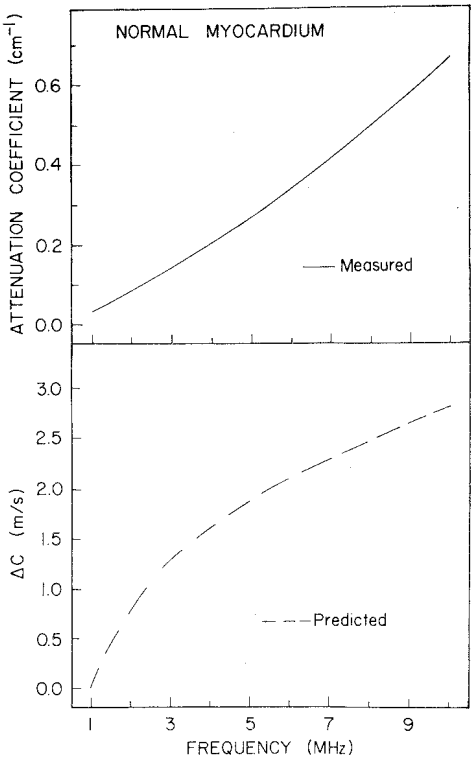

FIG. 5. The attenuation measured in normal dog myocardium is illustrated in the top panel. The lower panel presents the dispersion predicted by applying Eq. (30) to the attenuation data of the top panel.

nearly local, approximate relations was tested in a number of acoustic systems possessing a range of loss mechanisms. The results of theoretical analysis and experiments presented in Figs. 2, 3, and 4 clearly show that the approximate relations represent an accurate description of acoustic propagation in several systems which do not exhibit rapid variations with frequency over the range of interest.

The approximate expressions presented in Eq. (30) can be used to predict the ultrasonic properties of materials which may be difficult to obtain using currently available measurement techniques. For example, the ultrasonic dispersion is difficult to measure accurately in soft tissue specimens. However, the dispersion in soft tissue can be estimated from a knowledge of the measured attenuation coefficient. In Fig. 5 we predict the dispersion in normal dog myocardium from measurements of the attenuation coefficient obtained in our laboratory. The top panel of Fig. 5 illustrates the attenuation coefficient as a function of frequency over the range 1–10 MHz. In the lower panel of Fig. 5 we present the dispersion predicted by Eq. (30). According to Eq. (30), the nearly linear dependence of the attenuation coefficient on frequency for normal dog myocardium gives rise to the prediction of nearly logarithmic frequency dependence for the dispersion. The numerical value of the change in phase velocity from 1 to 10 MHz is less than 2 parts in 10^3 of the velocity. Consequently, the dispersion anticipated in normal dog myocardium, and most soft tissues, is very small. Therefore dispersive effects, such as distortion in the ultrasonic pulse shape, are correspondingly negligible.

Finally, as noted above, the relationship between the attenuation and dispersion is independent of the specific mechanism responsible for the attenuation. Several authors have stated that the existence of dispersion in a material is a strong indication that relaxational processes are responsible for the attenuation in that material.[13,14] As we noted in a previous publication,[6] and have clearly demonstrated in the present study, the existence of dispersion is in no way indicative of a particular mechanism of propagation, but rather merely establishes that a system which exhibits attenuation must exhibit dispersion if it satisfies the conditions of linearity and causality. An appreciation of which features of ultrasonic propagation are determined by general laws of physics as opposed to those features which are specific to the particular mechanism should prove useful in establishing the mechanisms responsible for the propagation of ultrasound in a material.

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