

Bayesian Analysis Users Guide
Release 4.00, Manual Version 1

G. Larry Bretthorst
Biomedical MR Laboratory
Washington University School Of Medicine,
Campus Box 8227
Room 2313, East Bldg.,
4525 Scott Ave.
St. Louis MO 63110
<http://bayes.wustl.edu>
Email: larry@bayes.wustl.edu

October 21, 2016

Contents

Manual Status	16
1 An Overview Of The Bayesian Analysis Software	19
1.1 The Server Software	19
1.2 The Client Interface	22
1.2.1 The Global Pull Down Menus	24
1.2.2 The Package Interface	24
1.2.3 The Viewers	27
2 Installing the Software	29
3 the Client Interface	33
3.1 The Global Pull Down Menus	35
3.1.1 the Files menu	35
3.1.2 the Packages menu	40
3.1.3 the WorkDir menu	45
3.1.4 the Settings menu	46
3.1.5 the Utilities menu	50
3.1.6 the Help menu	50
3.2 The Submit Job To Server area	51
3.3 The Server area	52
3.4 Interface Viewers	52
3.4.1 the Ascii Data Viewer	53
3.4.2 the fid Data Viewer	53
3.4.3 Image Viewer	59
3.4.3.1 the Image List area	59
3.4.3.2 the Set Image area	62
3.4.3.3 the Image Viewing area	62
3.4.3.4 the Grayscale area on the bottom	63
3.4.3.5 the Pixel Info area	63
3.4.3.6 the Image Statistics area	64
3.4.4 Prior Viewer	65
3.4.5 Fid Model Viewer	68
3.4.5.1 The fid Model Format	70

3.4.5.2	The Fid Model Reports	71
3.4.6	Plot Results Viewer	71
3.4.7	Text Results Viewer	74
3.4.8	Files Viewer	80
3.5	Common Interface Plots	80
3.5.1	Data, Model And Residual Plot	81
3.5.2	Posterior Probability For A Parameter	82
3.5.3	Maximum Entropy Histograms	83
3.5.4	Markov Monte Carlo Samples	83
3.5.5	Probability Vs Parameter Samples plot	86
3.5.6	Expected Log Likelihood Plot	88
3.5.7	Scatter Plots	88
3.5.8	Logarithm of the Posterior Probability Plot	91
3.5.9	Fortran/C Code Viewer	91
3.5.9.1	Fortran/C Model Viewer Popup Editor	94
4	An Introduction to Bayesian Probability Theory	99
4.1	The Rules of Probability Theory	99
4.2	Assigning Probabilities	102
4.3	Example: Parameter Estimation	109
4.3.1	Define The Problem	110
4.3.1.1	The Discrete Fourier Transform	110
4.3.1.2	Aliases	113
4.3.2	State The Model—Single-Frequency Estimation	114
4.3.3	Apply Probability Theory	115
4.3.4	Assign The Probabilities	118
4.3.5	Evaluate The Sums and Integrals	120
4.3.6	How Probability Generalizes The Discrete Fourier Transform	123
4.3.7	Aliasing	126
4.3.8	Parameter Estimates	132
4.4	Summary and Conclusions	136
5	Given Exponential Model	137
5.1	The Bayesian Calculation	139
5.2	Outputs From The Given Exponential Package	141
6	Unknown Number of Exponentials	143
6.1	The Bayesian Calculations	145
6.2	Outputs From The Unknown Number of Exponentials Package	148
7	Inversion Recovery	151
7.1	The Bayesian Calculation	153
7.2	Outputs From The Inversion Recovery Package	154

8	Bayes Analyze	155
8.1	Bayes Model	159
8.2	The Bayes Analyze Model Equation	161
8.3	The Bayesian Calculations	167
8.4	Levenberg-Marquardt And Newton-Raphson	171
8.5	Outputs From The Bayes Analyze Package	176
8.5.1	The “bayes.params.nnnn” Files	177
8.5.1.1	The Bayes Analyze File Header	178
8.5.1.2	The Global Parameters	182
8.5.1.3	The Model Components	184
8.5.2	The “bayes.model.nnnn” Files	185
8.5.3	The “bayes.output.nnnn” File	186
8.5.4	The “bayes.probabilities.nnnn” File	190
8.5.5	The “bayes.log.nnnn” File	193
8.5.6	The “bayes.status.nnnn” and “bayes.accepted.nnnn” Files	196
8.5.7	The “bayes.model.nnnn” File	197
8.5.8	The “bayes.summary1.nnnn” File	198
8.5.9	The “bayes.summary2.nnnn” File	199
8.5.10	The “bayes.summary3.nnnn” File	200
8.6	Bayes Analyze Error Messages	200
9	Big Peak/Little Peak	207
9.1	The Bayesian Calculation	209
9.2	Outputs From The Big Peak/Little Peak Package	216
10	Metabolic Analysis	219
10.1	The Metabolic Model	223
10.2	The Bayesian Calculation	225
10.3	The Metabolite Models	228
10.3.1	The IPGD_D2O Metabolite	228
10.3.2	The Glutamate.2.0 Metabolite	232
10.3.3	The Glutamate.3.0 Metabolite	235
10.4	The Example Metabolite	236
10.5	Outputs From The Bayes Metabolite Package	238
11	Find Resonances	239
11.1	The Bayesian Calculations	241
11.2	Outputs From The Bayes Find Resonances Package	246
12	Diffusion Tensor Analysis	247
12.1	The Bayesian Calculation	249
12.2	Using The Package	254
13	Big Magnetization Transfer	259
13.1	The Bayesian Calculation	259
13.2	Outputs From The Big Magnetization Transfer Package	262

14 Magnetization Transfer	265
14.1 The Bayesian Calculation	267
14.2 Using The Package	271
15 Magnetization Transfer Kinetics	275
15.1 The Bayesian Calculation	277
15.2 Using The Package	281
16 Given Polynomial Order	285
16.1 The Bayesian Calculation	287
16.1.1 Gram-Schmidt	287
16.1.2 The Bayesian Calculation	288
16.2 Outputs From the Given Polynomial Order Package	290
17 Unknown Polynomial Order	293
17.1 Bayesian Calculations	295
17.1.1 Assigning Priors	296
17.1.2 Assigning The Joint Posterior Probability	297
17.2 Outputs From the Unknown Polynomial Order Package	299
18 Errors In Variables	303
18.1 The Bayesian Calculation	305
18.2 Outputs From The Errors In Variables Package	308
19 Behrens-Fisher	311
19.1 Bayesian Calculation	311
19.1.1 The Four Model Selection Probabilities	314
19.1.1.1 The Means And Variances Are The Same	315
19.1.1.2 The Mean Are The Same And The Variances Differ	317
19.1.1.3 The Means Differ And The Variances Are The Same	318
19.1.1.4 The Means And Variances Differ	319
19.1.2 The Derived Probabilities	320
19.1.3 Parameter Estimation	321
19.2 Outputs From Behrens-Fisher Package	322
20 Enter Ascii Model	329
20.1 The Bayesian Calculation	331
20.1.1 The Bayesian Calculations Using Eq. (20.1)	331
20.1.2 The Bayesian Calculations Using Eq. (20.2)	332
20.2 Outputs Form The Enter Ascii Model Package	335
21 Enter Ascii Model Selection	337
21.1 The Bayesian Calculations	339
21.1.1 The Direct Probability With No Amplitude Marginalization	340
21.1.2 The Direct Probability With Amplitude Marginalization	342
21.1.2.1 Marginalizing the Amplitudes	343
21.1.2.2 Marginalizing The Noise Standard Deviation	348

21.2	Outputs Form The Enter Ascii Model Package	349
26	Phasing An Image	395
26.1	The Bayesian Calculation	396
26.2	Using The Package	402
27	Phasing An Image Using Non-Linear Phases	405
27.1	The Model Equation	405
27.2	The Bayesian Calculations	407
27.3	The Interfaces To The Nonlinear Phasing Routine	409
28	Analyze Image Pixel	411
28.1	Modification History	413
29	The Image Model Selection Package	415
29.1	The Bayesian Calculations	417
29.2	Outputs Form The Image Model Selection Package	418
A	Ascii Data File Formats	423
A.1	Ascii Input Data Files	423
A.2	Ascii Image File Formats	424
A.3	The Abscissa File Format	425
B	Markov chain Monte Carlo With Simulated Annealing	439
B.1	Metropolis-Hastings Algorithm	440
B.2	Multiple Simulations	441
B.3	Simulated Annealing	442
B.4	The Annealing Schedule	442
B.5	Killing Simulations	443
B.6	the Proposal	444
C	Thermodynamic Integration	445
D	McMC Values Report	449
E	Writing Fortran/C Models	455
E.1	Model Subroutines, No Marginalization	455
E.2	The Parameter File	458
E.3	The Subroutine Interface	460
E.4	The Subroutine Declarations	462
E.5	The Subroutine Body	463
E.6	Model Subroutines With Marginalization	464
F	the Bayes Directory Organization	469
G	4dfp Overview	471

8

H Outlier Detection

475

Bibliography

479

List of Figures

1.1	The Start Up Window	23
1.2	Example Package Exponential Interface	25
2.1	Installation Kit For The Bayesian Analysis Software	31
3.1	The Start Up Window	34
3.2	The Files Menu	35
3.3	The Files/Load Image Submenu	37
3.4	The Packages Menu	41
3.5	The Working Directory Menu	46
3.6	The Working Directory Information Popup	47
3.7	The Settings Pull Down Menu	47
3.8	The McMC Parameters Popup	48
3.9	The Edit Server Popup	49
3.10	The Submit Job Widgets	51
3.11	The Server Widgets Group	52
3.12	The Ascii Data Viewer	54
3.13	The Fid Data Viewer	55
3.14	Fid Data Display Type	56
3.15	Fid Data Options Menu	58
3.16	The Image Viewer	60
3.17	The Image Viewer Right Mouse Popup Menu	61
3.18	The Prior Probability Viewer	66
3.19	The Fid Model Viewer	69
3.20	The Plot Results Viewer	72
3.21	Plot Information Popup	73
3.22	The Text Results Viewer	75
3.23	The Bayes Condensed File	78
3.24	Data, Model, And Resid Plot	81
3.25	The Parameter Posterior Probabilities	82
3.26	The Maximum Entropy Histograms	84
3.27	The Parameter Samples Plot	85
3.28	Posterior Probability Vs Parameter Value	86
3.29	Posterior Probability Vs Parameter Value, A Skewed Example	87
3.30	The Expected Value Of The Logarithm Of The Likelihood	89

3.31	The Scatter Plots	90
3.32	The Logarithm Of The Posterior Probability By Repeat Plot	92
3.33	The Fortran/C Model Viewer	93
3.34	The Fortran/C Code Editor	95
4.1	Frequency Estimation Using The DFT	112
4.2	Aliases	113
4.3	Nonuniformly Nonsimultaneously Sampled Sinusoid	127
4.4	Alias Spacing	128
4.5	Which Is The Critical Time	130
4.6	Example, Frequency Estimation	131
4.7	Estimating The Sinusoids Parameters	133
5.1	The Given And Unknown Number Of Exponential Package Interface	138
6.1	The Unknown Exponential Interface	144
6.2	The Distribution Of Models	149
6.3	The Posterior Probability For Exponential Model	150
7.1	The Inversion Recovery Interface	152
8.1	Bayes Analyze Interface	156
8.2	Bayes Analyze Fid Model Viewer	160
8.3	The Bayes Analyze File Header	179
8.4	The bayes.noise File	180
8.5	Bayes Analyze Global Parameters	183
8.6	The Third Section Of The Parameter File	184
8.7	Example Of An Initial Model In The Output File	187
8.8	Base 10 Logarithm Of The Odds	187
8.9	A Small Sample Of The Output Report	188
8.10	Bayes Analyze Uncorrelated Output	189
8.11	The bayes.proBABILITIES.nnnn File	191
8.12	The bayes.log.nnnn File	193
8.13	The bayes.status.nnnn File	196
8.14	The bayes.model.nnnn File	197
8.15	The bayes.model.nnnn File Uncorrelated Resonances	198
8.16	Bayes Analyze Summary Header	198
8.17	The Summary2 (Best Summary)	199
8.18	The Summary3 Report	201
9.1	The Big Peak/Little Peak Interface	208
9.2	The Time Dependent Parameters	218
10.1	The Bayes Metabolite Interface	220
10.2	The Bayes Metabolite Viewer	222
10.3	Bayes Metabolite Parameters And Probabilities List	227
10.4	The IPGD_D20 Metabolite	229

10.5	Bayes Metabolite IPGD_D20 Spectrum	230
10.6	Bayes Metabolite, The Fraction of Glucose	231
10.7	Glutamate Example Spectrum	233
10.8	Estimating The F_{c0} , y and F_{a0} Parameters	236
10.9	Bayes Metabolite, The Ethyl Ether Example	237
11.1	The Find Resonances Interface With The Ethyl Ether Spectrum	240
12.1	The Diffusion Tensor Package Interface	248
12.2	Diffusion Tensor Parameter Estimates	256
12.3	Diffusion Tensor Posterior Probability For The Model	257
13.1	The Big Magnetization Package Interface	260
13.2	Big Magnetization Transfer Example Fid	263
13.3	Big Magnetization Transfer Expansion	263
13.4	Big Magnetization Transfer Peak Pick	264
14.1	The Magnetization Transfer Package Interface	266
14.2	Magnetization Transfer Package Peak Picking	272
14.3	Magnetization Transfer Example Data	273
14.4	Magnetization Transfer Example Spectrum	274
15.1	Magnetization Transfer Kinetics Package Interface	276
15.2	Magnetization Transfer Kinetics Package Arrhenius Plot	282
15.3	Magnetization Transfer Kinetics Water Viscosity Table	283
16.1	Given Polynomial Order Package Interface	286
16.2	Given Polynomial Order Scatter Plot	291
17.1	Unknown Polynomial Order Package Interface	294
17.2	The Distribution of Models On The Console Log	298
17.3	The Posterior Probability For The Polynomial Order	300
18.1	The Errors In Variables Package Interface	304
18.2	The McMC Values File Produced By The Errors In Variables Package	310
19.1	The Behrens-Fisher Interface	312
19.2	Behrens-Fisher Hypotheses Tested	313
19.3	Behrens-Fisher Console Log	323
19.4	Behrens-Fisher Status Listing	324
19.5	Behrens-Fisher McMC Values File, The Preamble	325
19.6	Behrens-Fisher McMC Values File, The Middle	326
19.7	Behrens-Fisher McMC Values File, The End	327
20.1	Enter Ascii Model Package Interface	330
21.1	The Enter Ascii Model Selection Package Interface	338

26.1	Absorption Model Images	396
26.2	The Interface To The Image Phasing Package	397
26.3	Linear Phasing Package The Console Log	403
27.1	Nonlinear Phasing Example	406
27.2	The Interface To The Nonlinear Phasing Package	410
28.1	The Interface To The Analyze Image Pixels Package	412
29.1	The Interface To The Image Model Selection Package	416
29.2	Single Exponential Example Image	419
29.3	Single Exponential Example Data	420
29.4	Posterior Probability For The ExpOneNoConst Model	421
A.1	Ascii Data File Format	424
D.1	The McMC Values Report Header	450
D.2	McMC Values Report, The Middle	451
D.3	The McMC Values Report, The End	452
E.1	Writing Models A Fortran Example	456
E.2	Writing Models A C Example	457
E.3	Writing Models, The Parameter File	459
E.4	Writing Models Fortran Declarations	463
E.5	Writing Models Fortran Example	466
E.6	Writing Models The Parameter File	467
G.1	Example FDF File Header	473
H.1	The Posterior Probability For The Number of Outliers	476
H.2	The Data, Model and Residual Plot With Outliers	478

List of Tables

8.1	Multiplet Relative Amplitudes	165
8.2	Bayes Analyze Models	181
8.3	Bayes Analyze Short Descriptions	195

Chapter 18

Errors In Variables

The “Errors in Variables” package fits polynomials to data when you have errors in both the abscissa X and the data value Y . The interface to this package is shown in Fig. 18.1. This interface is used to configure the Errors In Variables Package by setting both the polynomial order and by indicating what errors are known or given. Additionally, depending on the settings of the “Given Errors In” widget, the interface will load two, three or four column Ascii data. To use this package, you must do the following:

Select the Errors In Variables Package from the Package menu.

Using “Given Errors In” pull down menu select the type of Errors In Variables problem you wish to solve. Your choices are:

1. “Not Given” solves the Errors In Variables problem when the errors in both X and Y are not given. This option requires two column Ascii data, see Section 18.2 for a description of these files.
2. “ X and Y ” solves the Errors In Variables problem when the errors in X and Y are given. This option requires four column Ascii data.
3. “ X Only” solves the Errors In Variables problem when the errors in X are given. This option requires three column Ascii data.
4. “ Y Only” solves the Errors In Variables problem when the errors in Y are given. This option requires three column Ascii data.

Load one data set appropriate to the selected problem. Only “Not Given” uses standard two column Ascii data. Consequently, when this option is selected both Bayes Analyze and a Peak Pick can server as input. All of the other options require input of either three or four column Ascii files which can only be loaded using the “Files/Load Ascii/Files” menu. When you have successfully loaded a data set it will be plotted in the Ascii Data Viewer. However, the plot is a simple XY plot and no attempt is made by the interface to show the error bars in either X or Y .

Using the “Order” pull down menu, set the order of the polynomial to fit to the data.

Figure 18.1: The Errors In Variables Package Interface

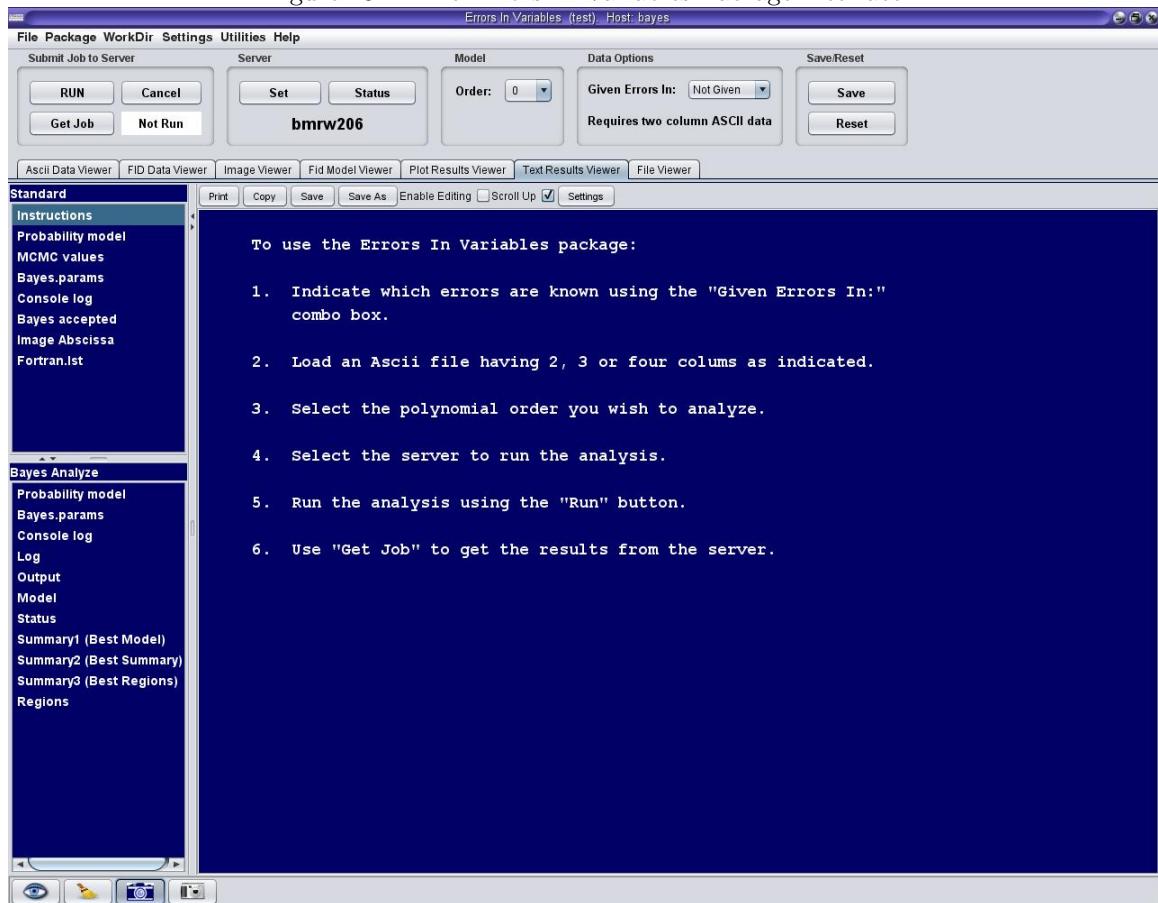


Figure 18.1: The Errors In Variables Package solves the problem of fitting polynomials when there are errors in both X and Y where X is an abscissa and Y is some function of X . Here this function is assumed to be a polynomial of a given order.

Select the server that is to process the analysis.

Check the status of the selected server to determine if the server is busy, change to another server if the selected server is busy.

Run the the analysis on the selected server by activating the Run button.

Get the the results of the analysis by activating the Get Job button. If the analysis is running, this button will return the Accepted report containing the status of the current run. Otherwise, it will fetch and display the results from the current analysis.

18.1 The Bayesian Calculation

The problem we are considering is one in which the data consists of measured pairs of numbers, (\hat{x}_i, \hat{y}_i) , where both the measured abscissa data value, \hat{x}_i , and the measured ordinate, \hat{y}_i , contain errors, i.e., noise, and we wish to fit a polynomial to these measured data pairs. If we designate the “true” but unknown values of (\hat{x}_i, \hat{y}_i) as (x_i, y_i) then the polynomial one would normally fit is given by

$$y = F(x) = \sum_{j=0}^m B_j H_j(x) \tag{18.1}$$

where the polynomial, $F(x)$, is evaluated at the true value of x giving a true value of y , B_j is the amplitude of the polynomial $H_j(x)$. These polynomials are the same polynomials discussed in Chapter 16 and are the Gram-Schmidt polynomials generated from x^j . Unfortunately, neither the y values nor the x values are known. An abscissa data value, \hat{x}_i , is related to the unknown true abscissa, x_i , by

$$\hat{x}_i = x_i + e_i \tag{18.2}$$

where e_i is the error in the measured abscissa. Similarly, a measured ordinate value, \hat{y}_i , is related to the unknown true ordinate, y_i , by

$$\hat{y}_i = y_i + n_i \tag{18.3}$$

where n_i is the error in the measured ordinate.

In the following calculation we will assume that the errors in the abscissa and ordinate are both unknown. We do this for the simple reason that it is the harder, i.e., more interesting calculation. However, in some cases one or both of these errors are actually known and the computer program that implements this package implements four different cases: 1) errors in both abscissa and ordinate known, 2) errors in the abscissa known but the ordinate error is unknown, etc.

If we Taylor expand the polynomial, $F(x)$, around \hat{x}_i , then

$$F(x_i) \approx F(\hat{x}_i) + F'(\hat{x}_i)(\hat{x}_i - x_i) \tag{18.4}$$

where

$$F'(\hat{x}_i) = \frac{dF(\hat{x}_i)}{d\hat{x}_i}. \tag{18.5}$$

This first order Taylor expansion will gives a reasonable approximation to $F(x)$ if the errors in the unknown abscissa are not large, i.e., if the polynomial is approximately linear over the likely error in

the abscissa. We make the Taylor expansion because, now the unknown true values of x_i appear in the model in a linear fashion and consequently we will be able to remove them by marginalization.

The probabilities we want to compute are the marginal posterior probabilities for the amplitudes of the polynomials and standard deviations of the noise prior probabilities. To calculate these, the numerical simulations must target the joint posterior probability for the amplitudes, $\{B_0, \dots, B_m\}$, and the noise standard deviations, σ_x and σ_y given the abscissa and ordinate data. This joint posterior probability is represented symbolically by $P(B_0 \cdots B_m \sigma_x \sigma_y | \hat{x} \hat{y} I)$, where \hat{x} and \hat{y} represent the abscissa and ordinate data.

The joint posterior probability for the amplitudes and the noise standard deviations targeted by the Markov chain Monte Carlo simulation, $P(B_0 \cdots B_m \sigma_x \sigma_y | \hat{x} \hat{y} I)$, is a marginal probability:

$$P(B_0 \cdots B_m \sigma_x \sigma_y | \hat{x} \hat{y} I) = \int P(B_0 \cdots B_m \sigma_x \sigma_y \{x\} | \hat{x} \hat{y} I) d\{x\} \quad (18.6)$$

where the integrals are over all of the unknown abscissa values. To compute this posterior probability, one factors the right-hand side of this equation using Bayes' theorem and the product rule to obtain

$$\begin{aligned} P(B_0 \cdots B_m \sigma_x \sigma_y \{x\} | \hat{x} \hat{y} I) &\propto P(\sigma_x | I) P(\sigma_y | I) \prod_{j=0}^m P(B_j | I) \\ &\times \prod_{i=1}^N P(x_i | I) \\ &\times \prod_{i=1}^N P(\hat{x}_i | x_i \sigma_x I) \\ &\times \prod_{i=1}^N P(\hat{y}_i | x_i B_0 \cdots B_m \sigma_y I). \end{aligned} \quad (18.7)$$

The prior probability for the standard deviation of the noise for the abscissa data, $P(\sigma_x | I)$, will be assigned as a bound Jeffreys' prior

$$P(\sigma_x | I) = \begin{cases} \frac{1}{R_x \sigma_x} & \text{if } \sigma_{xL} \leq \sigma_x \leq \sigma_{xH} \\ 0 & \text{otherwise} \end{cases} \quad (18.8)$$

where σ_{xL} and σ_{xH} are a lower and upper bound on σ_x . The normalization constant R_x is given by

$$R_x = \int_{\sigma_{xL}}^{\sigma_{xH}} \frac{d\sigma_x}{\sigma_x} = \log(\sigma_{xH}/\sigma_{xL}). \quad (18.9)$$

The bounds, σ_{xL} and σ_{xH} , are set rather pragmatically within the program that implements this package. The program computes the mean-square deviation, $\langle \bar{x}^2 - \bar{x}^2 \rangle$, using the \hat{x} data, and then sets the lower and upper bounds

$$\sigma_{xL} = \frac{\langle \bar{x}^2 - \bar{x}^2 \rangle}{10} \quad \text{and} \quad \sigma_{xH} = 10 \langle \bar{x}^2 - \bar{x}^2 \rangle, \quad (18.10)$$

which restricts σ_x to a two order of magnitude variation. Similarly, $P(\sigma_y | I)$, will be assigned

$$P(\sigma_y | I) = \begin{cases} \frac{1}{R_y \sigma_y} & \text{if } \sigma_{yL} \leq \sigma_y \leq \sigma_{yH} \\ 0 & \text{otherwise} \end{cases} \quad (18.11)$$

where the normalization constant, R_y , and the lower and upper bounds are computed analogously to corresponding abscissa values. The prior probability for the amplitudes, $P(B_j|I)$, is assigned a bounded zero mean Gaussian. The bounds are set at plus and minus ten times the largest projection of the model onto the data:

$$P(B_j|I) = \begin{cases} (2\pi\sigma_B^2)^{-1/2} \exp\left\{-\frac{B_j^2}{2\sigma_B^2}\right\} & \text{if } B_L \leq B_j \leq B_H \\ 0 & \text{otherwise} \end{cases} \quad (18.12)$$

with $B_L = -B_H$ and $B_H = \max(\text{abs}[\sum_{i=1}^N \hat{y}_i H_j(\hat{x}_i)])$. The standard deviation of this prior is $\sigma_B = B_H/3$. So the interval, $B_H - B_L$, represents a 6 standard deviation interval and the prior serves as little more than a way to keep the amplitudes from wandering into an unphysical region of the parameter space.

In this problem the unknown true values, the x_i , are parameters and one must assign a prior probability to all such parameters. The prior probability for the x_i , the $P(x_i|I)$, are assigned as unbounded Gaussians having mean equal to \tilde{x}_i and standard deviation, σ_p :

$$P(x_i|I) = (2\pi\sigma_p^2)^{-1/2} \exp\left\{-\frac{(\tilde{x}_i - x_i)^2}{2\sigma_p^2}\right\} \quad (18.13)$$

where \tilde{x}_i is the sampling point we thought we were measuring, and we set σ_p equal to the average x interval. We did this because it greatly simplified the formulas that must be programmed and so makes for a faster program, without changing the results to within the error bars.

The likelihood for a measured \hat{x}_i data value, $P(\hat{x}_i|x_i\sigma_x I)$, was assigned a Gaussian

$$P(\hat{x}_i|x_i\sigma_x I) = (2\pi\sigma_x^2)^{-1/2} \exp\left\{-\frac{(\hat{x}_i - x_i)^2}{2\sigma_x^2}\right\} \quad (18.14)$$

and the likelihood for a measured \hat{y}_i data value, $P(\hat{y}_i|\{A\}\sigma_y I)$, was assigned a Gaussian likelihood of the form:

$$P(\hat{y}_i|B_0 \cdots B_m \sigma_y I) = (2\pi\sigma_y^2)^{-1/2} \exp\left\{-\frac{(\hat{y}_i - F(\hat{x}_i) - F'(\hat{x}_i)[\hat{x}_i - x_i])^2}{2\sigma_y^2}\right\}. \quad (18.15)$$

If we now accumulate all of the priors, Eqs. (18.8,18.11,18.13), and likelihoods, Eqs.(18.14,18.15),

and substitute them into Eq. (18.6) one obtains

$$\begin{aligned}
P(B_0 \cdots B_m \sigma_x \sigma_y | \hat{x} \hat{y} I) &\propto \int_{-\infty}^{+\infty} dx_1 \cdots dx_N \frac{1}{R_x \sigma_x} \frac{1}{R_y \sigma_y} \\
&\times (2\pi \sigma_B^2)^{-\frac{m+1}{2}} \exp \left\{ -\sum_{j=0}^m \frac{B_j^2}{2\sigma_B^2} \right\} \\
&\times (2\pi \sigma_p^2)^{-\frac{N}{2}} \exp \left\{ -\sum_{i=1}^N \frac{(\tilde{x}_i - x_i)^2}{2\sigma_p^2} \right\} \\
&\times (2\pi \sigma_x^2)^{-\frac{N}{2}} \exp \left\{ -\sum_{i=1}^N \frac{(\hat{x}_i - x_i)^2}{2\sigma_x^2} \right\} \\
&\times (2\pi \sigma_y^2)^{-\frac{N}{2}} \exp \left\{ -\sum_{i=1}^N \frac{(\hat{y}_i - F(\hat{x}_i) - F'(\hat{x}_i)[\hat{x}_i - x_i])^2}{2\sigma_y^2} \right\}
\end{aligned} \tag{18.16}$$

as the posterior probability for the amplitudes and noise standard deviations. Evaluating the N integrals over the x_i , one obtains

$$P(B_0 \cdots B_m \sigma_x \sigma_y | \hat{x} \hat{y}) \propto \frac{1}{\sigma_x \sigma_y} \exp \left\{ -\sum_{j=0}^m \frac{B_j^2}{2\sigma_B^2} \right\} \prod_{i=1}^N \left(\frac{\sigma_x \sigma_y}{\sigma_i} \right) \exp \left\{ -\frac{Q_i}{2\sigma_i^2} \right\} \tag{18.17}$$

where we have dropped some constants that cancel when this probability density function is normalized. The function, Q_i , is given by

$$Q_i \equiv [\hat{y}_i - F(\hat{x}_i)]^2 [\sigma_p^2 + \sigma_x^2] - 2F'(\hat{x}_i) \sigma_x^2 [\hat{x}_i - \tilde{x}_i] [\hat{y}_i - F(\hat{x}_i)] + [F'(\hat{x}_i)^2 \sigma_x^2 + \sigma_y^2] [\hat{x}_i - \tilde{x}_i]^2 \tag{18.18}$$

where

$$\sigma_i \equiv \sqrt{\sigma_x^2 \sigma_y^2 + \sigma_p^2 \sigma_y^2 + F'(\hat{x}_i) \sigma_p^2 \sigma_x^2}. \tag{18.19}$$

In the special case that the Errors In Variables package implements, $\tilde{x}_i = \hat{x}_i$, Q_i simplifies and one obtains

$$P(B_0 \cdots B_m \sigma_x \sigma_y | \hat{x} \hat{y}) \propto \frac{1}{\sigma_x \sigma_y} \exp \left\{ -\sum_{j=0}^m \frac{B_j^2}{2\sigma_B^2} \right\} \prod_{i=1}^N \left(\frac{\sigma_x \sigma_y}{\sigma_i} \right) \exp \left\{ -\frac{[\hat{y}_i - F(\hat{x}_i)]^2 [\sigma_p^2 + \sigma_x^2]}{2\sigma_i^2} \right\}. \tag{18.20}$$

Computationally, this special case is simpler to calculate, so the program runs faster without giving up the ability to estimate both σ_x and σ_y and it is this probability density function that is targeted by the Markov chain Monte Carlo simulation.

18.2 Outputs From The Errors In Variables Package

The Text outputs files from the Errors In Variables packages consist of: “Bayes.prob.model,” “BayesErrInVarsGiven.mcmc.values,” “Bayes.params,” “Console.log,” “Bayes.accepted” and a “Bayes.Condensed.Fil

These output files can be viewed using the Text Viewer or they can be viewed using File Viewer by navigating to the current working directory and then selecting the files. The format of the `mcmc.values` report is discussed in Appendix D and the other reports are discussed in Chapter 3. Additionally, the “Plot Results Viewer” can be used to view the output probability density functions. In addition to the standard data, model and residual plots there are probability density functions for the decay rate constants, decay times, the amplitudes for each data set for each exponential and finally there are probability density functions for the standard deviation of the noise in each data set.

The only thing the least bit unusual about this package is the Ascii data that is required. In most Ascii packages the data are two columns, an abscissa and an ordinate. However, here there are four different file formats:

1. When the errors in both the abscissa and ordinate are unknown, two column Ascii data is required. Column one is the abscissa and column two is the ordinate. These data may be generated and or loaded using the files menu.
2. When the errors are known in the abscissa but not in the ordinate, three column Ascii data is required. Column one is the abscissa, column two is the ordinate, and column three is the error in the abscissa. This input can only be loaded using the “Files/Load Ascii/File” menu.
3. When the errors are known in the ordinate but not in the abscissa, three column Ascii data is required. Column one is the abscissa, column two is the ordinate, and column three is the error in the ordinate. This input can only be loaded using the “Files/Load Ascii/File” menu.
4. When the errors are known in both the abscissa and the ordinate, four column Ascii data is required. Column one is the abscissa, column two is the ordinate, column three is the error in the abscissa and column four is the error in the ordinate. This input can only be loaded using the “Files/Load Ascii/File” menu.

There are four test data sets located in the “`Bayes.test.data/ErrorsInVariables`” directory that may be used for testing. These four data sets are named: `ErrInVar_given_x.dat`, `ErrInVar_given_xy.dat`, `ErrInVar_given_y.dat`, and `ErrInVar_not_given.dat` respectively. An example of the MCMC values report generated by the Errors In Variables package is shown in Fig. 18.2. This report was generated using the “`ErrInVar_given_xy.dat`” data. The top section of the report contains some configuration information followed by information about the priors. This is followed by some averages over the various probabilities including the posterior probability for the model. This is followed by the parameters that maximized the joint marginal posterior probability for the parameters. Finally, the mean and standard deviation estimates of the amplitudes of the polynomials are given. For more on the general layout of the MCMC value file, see Appendix D.

In addition to the MCMC values file there are the standard Data, Model and Residual plots which can be viewed using the Plot Results Viewer. There are posterior probability density functions for the uncertainty in the abscissa data values, this plot is named “Sigma X” and there is a posterior probability density for the uncertainty in the ordinate, named “Sigma Y.” Additionally, there is one output probability density function for each amplitude in the polynomial being analyzed. So if you are analyzing a 6th order polynomial, there are seven output probability density functions. Finally, there are the output plots that come at the end of the plot list. These include the expected logarithm of the likelihood as a function of the annealing parameter, the scatter plots and the logarithm of the posterior probability for each simulation as a function of repeat number.

Figure 18.2: The MCMC Values File Produced By The Errors In Variables Package

The Parameter File Listing for the Errors in Variables Package

```

! BayesErrInVarsGiven Package
! Created 10-Feb-2012 10:11:27 by larry
!
      Output Dir = BayesOtherAnalysis
Number Of Abscissa = 1
Number Of Columns = 1
Number Of Sets = 1
      File Name = BayesOtherAnalysis/ErrInVar_given_xy.dat
MCMC Simulations = 48
      MCMC Repeats = 21
Minimum Annealing Steps = 21
      Histogram Type = Binned
Outlier Detection = Disabled
      Number Of Priors = 0
Package Parameters = 2
Total Mcmc Samples = 1008
      Kill Count = 4

```

MCMC Values Report for the Errors In Variables package

Param Desc	Priors Used In This Run				
	Low	Mean	High	Sigma	Norm
Coef 0.1	-1.504E+03	0.000E+00	1.504E+03	4.513E+02	-3.626E+00
Coef 1.1	-1.504E+03	0.000E+00	1.504E+03	4.513E+02	-3.626E+00
X 1.1	-5.445E-02	-4.785E-03	4.488E-02	9.933E-03	-3.222E+00
X 101.1	9.488E-01	9.985E-01	1.048E+00	9.933E-03	-3.222E+00

	Avg.	Sd.
The Average Log Posterior Probability:	194.0262	122.93507
The Average Log Prior Params:	-9.0257	0.00079
The Average Log Likelihood:	4.06109543E+02	1.37923E+00
The Log Posterior Probability For The Model:	3.74864004E+02	

The parameters that maximized the posterior probability are:

Parameter Description	Parameter
Std Dev in X	4.65331682E-04
Std Dev in Y	1.10193510E+00
Coef 0.1	1.01151226E+01
Coef 1.1	9.68326780E+00

The expected parameter values (mean value of the probability distributions):

Parameter Description	Mean Value	Std. Dev.	Peak Value
Sigma X	1.09472E-03	5.34846E-04	4.65332E-04
Sigma Y	1.09118E+00	7.44062E-02	1.10194E+00
Coef 1.0	1.01672E+01	2.01540E-01	1.01151E+01
Coef 1.1	9.59826E+00	3.49706E-01	9.68327E+00

Figure 18.2: This is the Errors In Variables mcmc.values file. It is the primary printed output from the Errors In Variables package. This report was generated using test data found in Bayes.test.data, the ErrorsInVariables subdirectory, the data file was ErrInVar_given_xy.dat. The top section of the report contains some configuration information followed by information about the priors. This is followed by some averages over the various probabilities including the posterior probability for the model. This is followed by the parameters that maximized the joint marginal posterior probability. Finally, the mean and standard deviation estimates of the amplitudes of the polynomials are given.

Bibliography

- [1] Rev. Thomas Bayes (1763), “An Essay Toward Solving a Problem in the Doctrine of Chances,” *Philos. Trans. R. Soc. London*, **53**, pp. 370-418; reprinted in *Biometrika*, **45**, pp. 293-315 (1958), and *Facsimiles of Two Papers by Bayes*, with commentary by W. Edwards Deming, New York, Hafner, 1963.
- [2] G. Larry Bretthorst (1988), “Bayesian Spectrum Analysis and Parameter Estimation,” in *Lecture Notes in Statistics*, **48**, J. Berger, S. Fienberg, J. Gani, K. Krickenberg, and B. Singer (eds), Springer-Verlag, New York, New York.
- [3] G. Larry Bretthorst (1990), “An Introduction to Parameter Estimation Using Bayesian Probability Theory,” in *Maximum Entropy and Bayesian Methods*, Dartmouth College 1989, P. Fougère ed., pp. 53-79, Kluwer Academic Publishers, Dordrecht the Netherlands.
- [4] G. Larry Bretthorst (1990), “Bayesian Analysis I. Parameter Estimation Using Quadrature NMR Models” *J. Magn. Reson.*, **88**, pp. 533-551.
- [5] G. Larry Bretthorst (1990), “Bayesian Analysis II. Signal Detection And Model Selection” *J. Magn. Reson.*, **88**, pp. 552-570.
- [6] G. Larry Bretthorst (1990), “Bayesian Analysis III. Examples Relevant to NMR” *J. Magn. Reson.*, **88**, pp. 571-595.
- [7] G. Larry Bretthorst (1991), “Bayesian Analysis. IV. Noise and Computing Time Considerations,” *J. Magn. Reson.*, **93**, pp. 369-394.
- [8] G. Larry Bretthorst (1992), “Bayesian Analysis. V. Amplitude Estimation for Multiple Well-Separated Sinusoids,” *J. Magn. Reson.*, **98**, pp. 501-523.
- [9] G. Larry Bretthorst (1992), “Estimating The Ratio Of Two Amplitudes In Nuclear Magnetic Resonance Data,” in *Maximum Entropy and Bayesian Methods*, C. R. Smith et al. (eds.), pp. 67-77, Kluwer Academic Publishers, the Netherlands.
- [10] G. Larry Bretthorst (1993), “On The Difference In Means,” in *Physics & Probability Essays in honor of Edwin T. Jaynes*, W. T. Grandy and P. W. Milonni (eds.), pp. 177-194, Cambridge University Press, England.
- [11] G. Larry Bretthorst (1996), “An Introduction To Model Selection Using Bayesian Probability Theory,” in *Maximum Entropy and Bayesian Methods*, G. R. Heidbreder, ed., pp. 1-42, Kluwer Academic Publishers, Printed in the Netherlands.

- [12] G. Larry Bretthorst (1999), "The Near-Irrelevance of Sampling Frequency Distributions," in *Maximum Entropy and Bayesian Methods*, W. von der Linden *et al.* (eds.), pp. 21-46, Kluwer Academic Publishers, the Netherlands.
- [13] G. Larry Bretthorst (2001), "Nonuniform Sampling: Bandwidth and Aliasing," in *Maximum Entropy and Bayesian Methods in Science and Engineering*, Joshua Rychert, Gary Erickson and C. Ray Smith *eds.*, pp. 1-28, American Institute of Physics, USA.
- [14] G. Larry Bretthorst, Christopher D. Kroenke, and Jeffrey J. Neil (2004), "Characterizing Water Diffusion In Fixed Baboon Brain," in *Bayesian Inference And Maximum Entropy Methods In Science And Engineering*, Rainer Fischer, Roland Preuss and Udo von Toussaint *eds.*, AIP conference Proceedings, **735**, pp. 3-15.
- [15] G. Larry Bretthorst, William C. Hutton, Joel R. Garbow, and Joseph J.H. Ackerman (2005), "Exponential parameter estimation (in NMR) using Bayesian probability theory," *Concepts in Magnetic Resonance*, 27A, Issue 2, pp. 55-63.
- [16] G. Larry Bretthorst, William C. Hutton, Joel R. Garbow, and Joseph J. H. Ackerman (2005), "Exponential model selection (in NMR) using Bayesian probability theory," *Concepts in Magnetic Resonance*, 27A, Issue 2, pp. 64-72.
- [17] G. Larry Bretthorst, William C. Hutton, Joel R. Garbow, and Joseph J.H. Ackerman (2005), "How accurately can parameters from exponential models be estimated? A Bayesian view," *Concepts in Magnetic Resonance*, 27A, Issue 2, pp. 73-83.
- [18] G. Larry Bretthorst, W. C. Hutton, J. R. Garbow, and Joseph J. H. Ackerman (2008), "High Dynamic Range MRS Time-Domain Signal Analysis," *Magn. Reson. in Med.*, **62**, pp. 1026-1035.
- [19] V. Chandramouli, K. Ekberg, W. C. Schumann, S. C. Kalhan, J. Wahren, and B. R. Landau (1997), "Quantifying gluconeogenesis during fasting," *American Journal of Physiology*, **273**, pp. H1209-H1215.
- [20] R. T. Cox (1961), "The Algebra of Probable Inference," Johns Hopkins Univ. Press, Baltimore.
- [21] André d'Avignon, G. Larry Bretthorst, Marlyn Emerson Holtzer, and Alfred Holtzer (1998), "Site-Specific Thermodynamics and Kinetics of a Coiled-Coil Transition by Spin Inversion Transfer NMR," *Biophysical Journal*, **74**, pp. 3190-3197.
- [22] André d'Avignon, G. Larry Bretthorst, Marlyn Emerson Holtzer, and Alfred Holtzer (1999), "Thermodynamics and Kinetics of a Folded-Folded Transition at Valine-9 of a GCN4-Like Leucine Zipper," *Biophysical Journal*, **76**, pp. 2752-2759.
- [23] David Freedman, and Persi Diaconis (1981), "On the histogram as a density estimator: L_2 theory," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, **57**, 4, pp. 453-476.
- [24] W. R. Gilks, S. Richardson, and D. J. Spiegelhalter (1996), "Markov Chain Monte Carlo in Practice," Chapman & Hall, London.

- [25] Paul M. Goggans, and Ying Chi (2004), “Using Thermodynamic Integration to Calculate the Posterior Probability in Bayesian Model Selection Problems,” in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering: 23rd International Workshop*, **707**, pp. 59-66.
- [26] Marlyn Emerson Holtzer, G. Larry Bretthorst, D. André d’Avignon, Ruth Hogue Angelette, Lisa Mints, and Alfred Holtzer (2001), “Temperature Dependence of the Folding and Unfolding Kinetics of the GCN4 Leucine Lipper via ^{13}C alpha-NMR,” *Biophysical Journal*, **80**, pp. 939-951.
- [27] E. T. Jaynes (1968), “Prior Probabilities,” *IEEE Transactions on Systems Science and Cybernetics*, SSC-4, pp. 227-241; reprinted in [30].
- [28] E. T. Jaynes (1978), “Where Do We Stand On Maximum Entropy?” in *The Maximum Entropy Formalism*, R. D. Levine and M. Tribus *Eds.*, pp. 15-118, Cambridge: MIT Press, Reprinted in [30].
- [29] E. T. Jaynes (1980), “Marginalization and Prior Probabilities,” in *Bayesian Analysis in Econometrics and Statistics*, A. Zellner *ed.*, North-Holland Publishing Company, Amsterdam; reprinted in [30].
- [30] E. T. Jaynes (1983), “Papers on Probability, Statistics and Statistical Physics,” a reprint collection, D. Reidel, Dordrecht the Netherlands; second edition Kluwer Academic Publishers, Dordrecht the Netherlands, 1989.
- [31] E. T. Jaynes (1957), “How Does the Brain do Plausible Reasoning?” unpublished Stanford University Microwave Laboratory Report No. 421; reprinted in *Maximum-Entropy and Bayesian Methods in Science and Engineering* **1**, pp. 1-24, G. J. Erickson and C. R. Smith *Eds.*, 1988.
- [32] E. T. Jaynes (2003), “Probability Theory—The Logic of Science,” edited by G. Larry Bretthorst, Cambridge University Press, Cambridge UK.
- [33] Sir Harold Jeffreys (1939), “Theory of Probability,” Oxford Univ. Press, London; Later editions, 1948, 1961.
- [34] John G. Jones, Michael A. Solomon, Suzanne M. Cole, A. Dean Sherry, and Craig R. Malloy (2001) “An integrated ^2H and ^{13}C NMR study of gluconeogenesis and TCA cycle flux in humans,” *American Journal of Physiology, Endocrinology, and Metabolism*, **281**, pp. H848-H856.
- [35] John Kotyk, N. G. Hoffman, W. C. Hutton, G. Larry Bretthorst, and J. J. H. Ackerman (1992), “Comparison of Fourier and Bayesian Analysis of NMR Signals. I. Well-Separated Resonances (The Single-Frequency Case),” *J. Magn. Reson.*, **98**, pp. 483–500.
- [36] Pierre Simon Laplace (1814), “A Philosophical Essay on Probabilities,” John Wiley & Sons, London, Chapman & Hall, Limited 1902. Translated from the 6th edition by F. W. Truscott and F. L. Emory.
- [37] N. Lartillot, and H. Philippe (2006), “Computing Bayes Factors Using Thermodynamic Integration,” *Systematic Biology*, **55** (2), pp. 195-207.

- [38] D. Le Bihan, and E. Breton (1985), “Imagerie de diffusion in-vivo par rsonance,” Comptes rendus de l’Acadmie des Sciences (Paris), **301** (15), pp. 1109-1112.
- [39] N. R. Lomb (1976), “Least-Squares Frequency Analysis of Unevenly Spaced Data,” *Astrophysical and Space Science*, **39**, pp. 447-462.
- [40] T. J. Loredo (1990), “From Laplace To SN 1987A: Bayesian Inference In Astrophysics,” in *Maximum Entropy and Bayesian Methods*, P. F. Fougere (ed), Kluwer Academic Publishers, Dordrecht, The Netherlands.
- [41] Craig R. Malloy, A. Dean Sherry, and Mark Jeffrey (1988), “Evaluation of Carbon Flux and Substrate Selection through Alternate Pathways Involving the Citric Acid Cycle of the Heart by ^{13}C NMR Spectroscopy,” *Journal of Biological Chemistry*, **263** (15), pp. 6964-6971.
- [42] Craig R. Malloy, Dean Sherry, and Mark Jeffrey (1990), “Analysis of tricarboxylic acid cycle of the heart using ^{13}C isotope isomers,” *American Journal of Physiology*, **259**, pp. H987-H995.
- [43] Lawrence R. Mead and Nikos Papanicolaou, “Maximum entropy in the problem of moments,” *J. Math. Phys.* **25**, 2404–2417 (1984).
- [44] K. Merboldt, Wolfgang Hanicke, and Jens Frahm (1969), “Self-diffusion NMR imaging using stimulated echoes,” *Journal of Magnetic Resonance*, **64** (3), pp. 479-486.
- [45] Nicholas Metropolis, Arianna W. Rosenbluth, Marshall N. Rosenbluth, Augusta H. Teller, and Edward Teller (1953), “Equation of State Calculations by Fast Computing Machines,” *Journal of Chemical Physics*. The previous link is to the Americain Institute of Physics and if you do not have access to Science Sitations you many not be able to retrieve this paper.
- [46] Radford M. Neal (1993), “Probabilistic Inference Using Markov Chain Monte Carlo Methods,” technical report CRG-TR-93-1, Dept. of Computer Science, University of Toronto.
- [47] Jeffrey J. Neil, and G. Larry Bretthorst (1993), “On the Use of Bayesian Probability Theory for Analysis of Exponential Decay Data: An Example Taken from Intravoxel Incoherent Motion Experiments,” *Magn. Reson. in Med.*, **29**, pp. 642–647.
- [48] H. Nyquist (1924), “Certain Factors Affecting Telegraph Speed,” *Bell System Technical Journal*, **3**, pp. 324-346.
- [49] H. Nyquist (1928), “Certain Topics in Telegraph Transmission Theory,” *Transactions AIEE*, **3**, pp. 617-644.
- [50] William H. Press, Saul A. Teukolsky, William T. Vetterling and Brian P. Flannery (1992), “Numerical Recipes The Art of Scientific Computing Second Edition,” Cambridge University Press, Cambridge UK.
- [51] Emanuel Parzen (1962), “On Estimation of a Probability Density Function and Mode,” *Annals of Mathematical Statistics* **33**, 1065–1076
- [52] Karl Pearson (1895), “Contributions to the Mathematical Theory of Evolution. II. Skew Variation in Homogeneous Material,” *Phil. Trans. R. Soc. A* **186**, 343–326.

- [53] Murray Rosenblatt, "Remarks on Some Nonparametric Estimates of a Density Function," *Annals of Mathematical Statistics* **27**, 832–837 (1956).
- [54] Jeffery D. Scargle (1981), "Studies in Astronomical Time Series Analysis I. Random Process In The Time Domain," *Astrophysical Journal Supplement Series*, **45**, pp. 1-71.
- [55] Jeffery D. Scargle (1982), "Studies in Astronomical Time Series Analysis II. Statistical Aspects of Spectral Analysis of Unevenly Sampled Data," *Astrophysical Journal*, **263**, pp. 835-853.
- [56] Jeffery D. Scargle (1989), "Studies in Astronomical Time Series Analysis. III. Fourier Transforms, Autocorrelation Functions, and Cross-correlation Functions of Unevenly Spaced Data," *Astrophysical Journal*, **343**, pp. 874-887.
- [57] Arthur Schuster (1905), "The Periodogram and its Optical Analogy," *Proceedings of the Royal Society of London*, **77**, p. 136-140.
- [58] Claude E. Shannon (1948), "A Mathematical Theory of Communication," *Bell Syst. Tech. J.*, **27**, pp. 379-423.
- [59] John E. Shore, and Rodney W. Johnson (1981), "Properties of cross-entropy minimization," *IEEE Trans. on Information Theory*, **IT-27**, No. 4, pp. 472-482.
- [60] John E. Shore and Rodney W. Johnson (1980), "Axiomatic derivation of the principle of maximum entropy and the principle of minimum cross-entropy," *IEEE Trans. on Information Theory*, **IT-26** (1), pp. 26-37.
- [61] Devinderjit Sivia, and John Skilling (2006), "Data Analysis: A Bayesian Tutorial," Oxford University Press, USA.
- [62] Edward O. Stejskal and Tanner, J. E. (1965), "Spin Diffusion Measurements: Spin Echoes in the Presence of a Time-Dependent Field Gradient." *Journal of Chemical Physics*, **42** (1), pp. 288-292.
- [63] D. G. Taylor and Bushell, M. C. (1985), "The spatial mapping of translational diffusion coefficients by the NMR imaging technique," *Physics in Medicine and Biology*, **30** (4), pp. 345-349.
- [64] Myron Tribus (1969), "Rational Descriptions, Decisions and Designs," Pergamon Press, Oxford.
- [65] P. M. Woodward (1953), "Probability and Information Theory, with Applications to Radar," McGraw-Hill, N. Y. Second edition (1987); R. E. Krieger Pub. Co., Malabar, Florida.
- [66] Arnold Zellner (1971), "An Introduction to Bayesian Inference in Econometrics," John Wiley and Sons, New York.