

Bayesian Analysis Users Guide  
Release 4.00, Manual Version 1

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## Chapter 7

# Inversion Recovery

The inversion recovery package is an example of a preloaded package. Preloaded packages are special cases of the Enter Ascii package, where the model is loaded at the time the package is started. In this example, the model is a single exponential plus a constant with the two amplitudes expressed as an initial and final amplitude. The input data files analyzed by this package are Ascii and may be input from Ascii files, a peak pick, a Bayes Analyze file or they may be loaded from an image pixel. When “Inversion Recovery” package button is activated, the interface window shown in Fig. 7.1 is displayed. To use this package, you must do the following:

**Select** the inversion recovery package from the package menu.

**Load** one or more Ascii data sets using the Files menu.

**Check** the find outliers box if you suspect outliers are present in the data.

**Review** the prior probability for the decay rate constant to make sure the prior assignment is correct.

**Select** the name of the server that is to process the request.

**Check** the status of the selected server to determine if the server is busy, change to another server if the selected server is busy.

**Run** the the analysis on the selected server by activating the Run button.

**Get** the the results of the analysis by activating the Get Job button. If the analysis is running, this button will return the Accepted report containing the status of the current run.

The inversion recovery package has only one package specific button, the Find Outliers button and it is should only be used when you think there might be an outlier in the data. Additionally, the prior viewer can be used to setup the prior probability for the decay rate constant.

Figure 7.1: The Inversion Recovery Interface

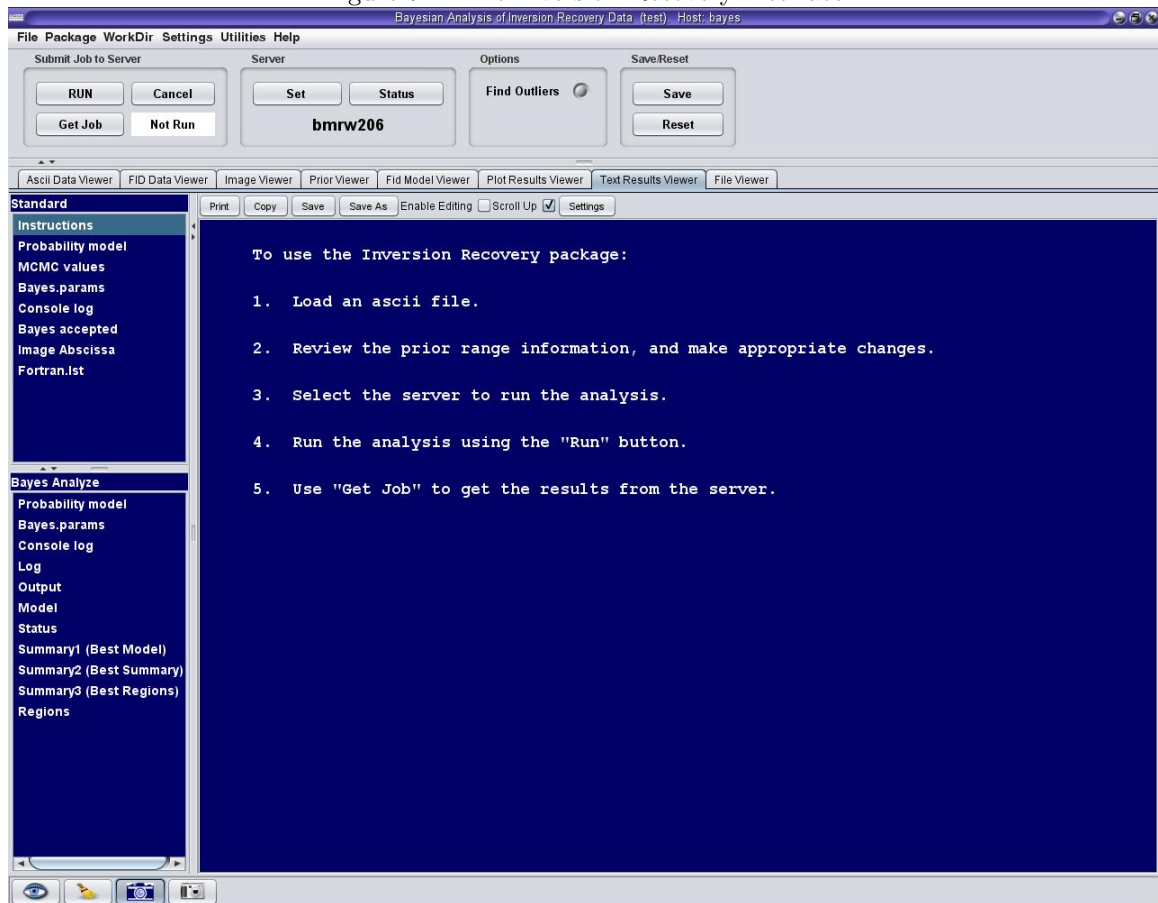


Figure 7.1: When the Inversion Recovery package is selected the interface shown here is displayed. The prior viewer can be used to set the prior probability for the decay rate constants. Note in this package the posterior probabilities are marginal posterior probabilities, so there is no prior for the amplitudes.

## 7.1 The Bayesian Calculation

The inversion recovery model is a single exponential model when the amplitudes are expressed as an initial amplitude in the  $j$ th data set,  $M_{0j}$  and a final amplitude  $M_{\infty j}$  in the  $j$ th data set:

$$d_{ij} = M_{\infty j} + (M_{0j} - M_{\infty j}) \exp\{-\alpha t_i\} + n_{ij} \quad (7.1)$$

where  $\alpha$  is the decay rate constant of the magnetization. It is easy to verify that  $M_{0j}$ , is the magnetization at  $t = 0$  by simply substituting  $t = 0$ , the model reduces to  $M_{0j}$  and one is estimating the initial magnetization. When  $t = \infty$ , the exponential goes to zero and one is estimating the ending magnetization,  $M_{\infty j}$  and the magnetization relaxes from  $M_{0j}$  to  $M_{\infty j}$ . Because of the way this model is written,  $M_{0j}$  is not forced to be negative,  $M_{0j}$  simply represents the beginning magnetization, while  $M_{\infty j}$  represents the ending magnetization. This package uses a marginal posterior probability for the decay rate constant, and consequently the integrals over the amplitudes are from minus to plus infinity.

The calculations are very standard, and proceeds by computing the marginal posterior probability for each parameter from the joint posterior probability for all of the parameters. So, for example, the posterior probability for the decay rate constant is given by

$$P(\alpha|DI) = \int P(\alpha\{M_0\}\{M_\infty\}\{\sigma\}|DI)d\{M_0\}d\{M_\infty\}d\{\sigma\} \quad (7.2)$$

where  $\{M_0\}$  is all of the initial amplitudes given the  $n$  input data sets, similarly  $\{M_\infty\}$  are all of the final amplitudes and  $\{\sigma\}$  are the standard deviations of the noise prior probabilities for the data sets. We assume that there is a single initial and final amplitude associated with each data set and that the data sets are independent, consequently, this equation can be factored using Bayes' Theorem [1] and the rules of probability theory

$$P(\alpha|DI) = \prod_{j=1}^n \int P(\alpha M_{0j} M_{\infty j} \sigma_j | D_j I) dM_{0j} dM_{\infty j} d\sigma_j \quad (7.3)$$

and factoring the joint posterior probability for all of the parameters one obtains:

$$P(\alpha|DI) = P(\alpha|I) \prod_{j=1}^n \int P(M_{0j}|I) P(M_{\infty j}|I) P(\sigma_j|I) P(D_j|\alpha M_{0j} M_{\infty j} \sigma_j I) dM_{0j} dM_{\infty j} d\sigma_j. \quad (7.4)$$

The prior probabilities for the amplitudes,  $P(M_{0j}|I)$  and  $P(M_{\infty j}|I)$  are assigned using broad Gaussian prior probability and to facilitate marginalization, these priors cannot be changed by the user. The prior probability for the standard deviations,  $P(\sigma_j|I)$ , are assigned using a Jeffreys' prior, [33]. The prior probabilities for the decay rate constant,  $P(\alpha|I)$ , can be set by the user. The integrals over both the magnetizations and the standard deviations are evaluated analytically using the techniques described in [2].

The posterior probability for the decay rate constant computed using this inversion recovery model is a marginal posterior probability. Consequently, only the decay rate constant occurs in the posterior probability. However, because the initial and final magnetizations were removed using marginalization, and the corresponding marginalization integrals were from minus to plus infinity, it is possible for this package to estimate an amplitude to be either positive or negative. If this is



in contradiction to the information available to the user, then using the Enter Ascii Model package with an inversion recovery model without marginalization will enable the user to impose any prior ranges needed on the initial and final magnetizations.

The full Bayesian calculation and the assignment of the prior probabilities is discussed in reference [15] and this paper is available in pdf by activating [this link](#). Additionally, much more about exponential parameter estimation is contained in [16, 17]. The [first](#) paper describes the problem of determining the number of exponentials in a given sample of data, while the [second](#) paper discusses how the accuracy of the parameter estimates depends on the number of data values, signal-to-noise level and the rate of decay of the sample.

## 7.2 Outputs From The Inversion Recovery Package

The Text outputs reports from the Inversion Recovery packages are very simple, consisting of a probability for the model file “Bayes.prob.model.” A file containing the results of the Markov chain Monte Carlo simulations “BayesEnterAscii.mcmc.Values” is written. This file contains the mean and standard deviation parameter estimates. A parameter file “Bayes.params” is generated. An output console log “Console.log” and an accepted report “Bayes.accepted,” are both written while the package is running. Finally a condensed file, “Bayes.Condensed.File,” is written that contains the parameter estimates in a condensed form that is more usable by a computer program. The format of the mcmc.values report is discussed in Appendix D and the other reports are discussed in Chapter 3. The “Plot Results Viewer” can be used to view the output probability density functions. In addition to the standard data, model and residual plots there are probability density functions for the decay rate constants, decay times, the initial and final magnetizations for each data set and finally there are probability density functions for the noise standard deviation in each data set.

# Bibliography

- [1] Rev. Thomas Bayes (1763), “An Essay Toward Solving a Problem in the Doctrine of Chances,” *Philos. Trans. R. Soc. London*, **53**, pp. 370-418; reprinted in *Biometrika*, **45**, pp. 293-315 (1958), and *Facsimiles of Two Papers by Bayes*, with commentary by W. Edwards Deming, New York, Hafner, 1963.
- [2] G. Larry Bretthorst (1988), “Bayesian Spectrum Analysis and Parameter Estimation,” in *Lecture Notes in Statistics*, **48**, J. Berger, S. Fienberg, J. Gani, K. Krickenberg, and B. Singer (eds), Springer-Verlag, New York, New York.
- [3] G. Larry Bretthorst (1990), “An Introduction to Parameter Estimation Using Bayesian Probability Theory,” in *Maximum Entropy and Bayesian Methods*, Dartmouth College 1989, P. Fougère ed., pp. 53-79, Kluwer Academic Publishers, Dordrecht the Netherlands.
- [4] G. Larry Bretthorst (1990), “Bayesian Analysis I. Parameter Estimation Using Quadrature NMR Models” *J. Magn. Reson.*, **88**, pp. 533-551.
- [5] G. Larry Bretthorst (1990), “Bayesian Analysis II. Signal Detection And Model Selection” *J. Magn. Reson.*, **88**, pp. 552-570.
- [6] G. Larry Bretthorst (1990), “Bayesian Analysis III. Examples Relevant to NMR” *J. Magn. Reson.*, **88**, pp. 571-595.
- [7] G. Larry Bretthorst (1991), “Bayesian Analysis. IV. Noise and Computing Time Considerations,” *J. Magn. Reson.*, **93**, pp. 369-394.
- [8] G. Larry Bretthorst (1992), “Bayesian Analysis. V. Amplitude Estimation for Multiple Well-Separated Sinusoids,” *J. Magn. Reson.*, **98**, pp. 501-523.
- [9] G. Larry Bretthorst (1992), “Estimating The Ratio Of Two Amplitudes In Nuclear Magnetic Resonance Data,” in *Maximum Entropy and Bayesian Methods*, C. R. Smith et al. (eds.), pp. 67-77, Kluwer Academic Publishers, the Netherlands.
- [10] G. Larry Bretthorst (1993), “On The Difference In Means,” in *Physics & Probability Essays in honor of Edwin T. Jaynes*, W. T. Grandy and P. W. Milonni (eds.), pp. 177-194, Cambridge University Press, England.
- [11] G. Larry Bretthorst (1996), “An Introduction To Model Selection Using Bayesian Probability Theory,” in *Maximum Entropy and Bayesian Methods*, G. R. Heidbreder, ed., pp. 1-42, Kluwer Academic Publishers, Printed in the Netherlands.

- [12] G. Larry Bretthorst (1999), “The Near-Irrelevance of Sampling Frequency Distributions,” in *Maximum Entropy and Bayesian Methods*, W. von der Linden *et al.* (eds.), pp. 21-46, Kluwer Academic Publishers, the Netherlands.
- [13] G. Larry Bretthorst (2001), “Nonuniform Sampling: Bandwidth and Aliasing,” in *Maximum Entropy and Bayesian Methods in Science and Engineering*, Joshua Rychert, Gary Erickson and C. Ray Smith *eds.*, pp. 1-28, American Institute of Physics, USA.
- [14] G. Larry Bretthorst, Christopher D. Kroenke, and Jeffrey J. Neil (2004), “Characterizing Water Diffusion In Fixed Baboon Brain,” in *Bayesian Inference And Maximum Entropy Methods In Science And Engineering*, Rainer Fischer, Roland Preuss and Udo von Toussaint *eds.*, AIP conference Proceedings, **735**, pp. 3-15.
- [15] G. Larry Bretthorst, William C. Hutton, Joel R. Garbow, and Joseph J.H. Ackerman (2005), “Exponential parameter estimation (in NMR) using Bayesian probability theory,” *Concepts in Magnetic Resonance*, 27A, Issue 2, pp. 55-63.
- [16] G. Larry Bretthorst, William C. Hutton, Joel R. Garbow, and Joseph J. H. Ackerman (2005), “Exponential model selection (in NMR) using Bayesian probability theory,” *Concepts in Magnetic Resonance*, 27A, Issue 2, pp. 64-72.
- [17] G. Larry Bretthorst, William C. Hutton, Joel R. Garbow, and Joseph J.H. Ackerman (2005), “How accurately can parameters from exponential models be estimated? A Bayesian view,” *Concepts in Magnetic Resonance*, 27A, Issue 2, pp. 73-83.
- [18] G. Larry Bretthorst, W. C. Hutton, J. R. Garbow, and Joseph J. H. Ackerman (2008), “High Dynamic Range MRS Time-Domain Signal Analysis,” *Magn. Reson. in Med.*, **62**, pp. 1026-1035.
- [19] V. Chandramouli, K. Ekberg, W. C. Schumann, S. C. Kalhan, J. Wahren, and B. R. Landau (1997), “Quantifying gluconeogenesis during fasting,” *American Journal of Physiology*, **273**, pp. H1209-H1215.
- [20] R. T. Cox (1961), “The Algebra of Probable Inference,” Johns Hopkins Univ. Press, Baltimore.
- [21] André d’Avignon, G. Larry Bretthorst, Marlyn Emerson Holtzer, and Alfred Holtzer (1998), “Site-Specific Thermodynamics and Kinetics of a Coiled-Coil Transition by Spin Inversion Transfer NMR,” *Biophysical Journal*, **74**, pp. 3190-3197.
- [22] André d’Avignon, G. Larry Bretthorst, Marlyn Emerson Holtzer, and Alfred Holtzer (1999), “Thermodynamics and Kinetics of a Folded-Folded Transition at Valine-9 of a GCN4-Like Leucine Zipper,” *Biophysical Journal*, **76**, pp. 2752-2759.
- [23] David Freedman, and Persi Diaconis (1981), “On the histogram as a density estimator:  $L_2$  theory,” *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, **57**, 4, pp. 453-476.
- [24] W. R. Gilks, S. Richardson, and D. J. Spiegelhalter (1996), “Markov Chain Monte Carlo in Practice,” Chapman & Hall, London.

- [25] Paul M. Goggans, and Ying Chi (2004), “Using Thermodynamic Integration to Calculate the Posterior Probability in Bayesian Model Selection Problems,” in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering: 23rd International Workshop*, **707**, pp. 59-66.
- [26] Marlyn Emerson Holtzer, G. Larry Bretthorst, D. André d’Avignon, Ruth Hogue Angelette, Lisa Mints, and Alfred Holtzer (2001), “Temperature Dependence of the Folding and Unfolding Kinetics of the GCN4 Leucine Lipper via  $^{13}\text{C}$  alpha-NMR,” *Biophysical Journal*, **80**, pp. 939-951.
- [27] E. T. Jaynes (1968), “Prior Probabilities,” *IEEE Transactions on Systems Science and Cybernetics*, SSC-4, pp. 227-241; reprinted in [30].
- [28] E. T. Jaynes (1978), “Where Do We Stand On Maximum Entropy?” in *The Maximum Entropy Formalism*, R. D. Levine and M. Tribus *Eds.*, pp. 15-118, Cambridge: MIT Press, Reprinted in [30].
- [29] E. T. Jaynes (1980), “Marginalization and Prior Probabilities,” in *Bayesian Analysis in Econometrics and Statistics*, A. Zellner *ed.*, North-Holland Publishing Company, Amsterdam; reprinted in [30].
- [30] E. T. Jaynes (1983), “Papers on Probability, Statistics and Statistical Physics,” a reprint collection, D. Reidel, Dordrecht the Netherlands; second edition Kluwer Academic Publishers, Dordrecht the Netherlands, 1989.
- [31] E. T. Jaynes (1957), “How Does the Brain do Plausible Reasoning?” unpublished Stanford University Microwave Laboratory Report No. 421; reprinted in *Maximum-Entropy and Bayesian Methods in Science and Engineering* **1**, pp. 1-24, G. J. Erickson and C. R. Smith *Eds.*, 1988.
- [32] E. T. Jaynes (2003), “Probability Theory—The Logic of Science,” edited by G. Larry Bretthorst, Cambridge University Press, Cambridge UK.
- [33] Sir Harold Jeffreys (1939), “Theory of Probability,” Oxford Univ. Press, London; Later editions, 1948, 1961.
- [34] John G. Jones, Michael A. Solomon, Suzanne M. Cole, A. Dean Sherry, and Craig R. Malloy (2001) “An integrated  $^2\text{H}$  and  $^{13}\text{C}$  NMR study of gluconeogenesis and TCA cycle flux in humans,” *American Journal of Physiology, Endocrinology, and Metabolism*, **281**, pp. H848-H856.
- [35] John Kotyk, N. G. Hoffman, W. C. Hutton, G. Larry Bretthorst, and J. J. H. Ackerman (1992), “Comparison of Fourier and Bayesian Analysis of NMR Signals. I. Well-Separated Resonances (The Single-Frequency Case),” *J. Magn. Reson.*, **98**, pp. 483–500.
- [36] Pierre Simon Laplace (1814), “A Philosophical Essay on Probabilities,” John Wiley & Sons, London, Chapman & Hall, Limited 1902. Translated from the 6th edition by F. W. Truscott and F. L. Emory.
- [37] N. Lartillot, and H. Philippe (2006), “Computing Bayes Factors Using Thermodynamic Integration,” *Systematic Biology*, **55** (2), pp. 195-207.

- [38] D. Le Bihan, and E. Breton (1985), “Imagerie de diffusion in-vivo par rsonance,” Comptes rendus de l’Acadmie des Sciences (Paris), **301** (15), pp. 1109-1112.
- [39] N. R. Lomb (1976), “Least-Squares Frequency Analysis of Unevenly Spaced Data,” *Astrophysical and Space Science*, **39**, pp. 447-462.
- [40] T. J. Loredo (1990), “From Laplace To SN 1987A: Bayesian Inference In Astrophysics,” in *Maximum Entropy and Bayesian Methods*, P. F. Fougere (ed), Kluwer Academic Publishers, Dordrecht, The Netherlands.
- [41] Craig R. Malloy, A. Dean Sherry, and Mark Jeffrey (1988), “Evaluation of Carbon Flux and Substrate Selection through Alternate Pathways Involving the Citric Acid Cycle of the Heart by  $^{13}\text{C}$  NMR Spectroscopy,” *Journal of Biological Chemistry*, **263** (15), pp. 6964-6971.
- [42] Craig R. Malloy, Dean Sherry, and Mark Jeffrey (1990), “Analysis of tricarboxylic acid cycle of the heart using  $^{13}\text{C}$  isotope isomers,” *American Journal of Physiology*, **259**, pp. H987-H995.
- [43] Lawrence R. Mead and Nikos Papanicolaou, “Maximum entropy in the problem of moments,” *J. Math. Phys.* **25**, 2404–2417 (1984).
- [44] K. Merboldt, Wolfgang Hanicke, and Jens Frahm (1969), “Self-diffusion NMR imaging using stimulated echoes,” *Journal of Magnetic Resonance*, **64** (3), pp. 479-486.
- [45] Nicholas Metropolis, Arianna W. Rosenbluth, Marshall N. Rosenbluth, Augusta H. Teller, and Edward Teller (1953), “Equation of State Calculations by Fast Computing Machines,” *Journal of Chemical Physics*. The previous link is to the Americain Institute of Physics and if you do not have access to Science Sitations you many not be able to retrieve this paper.
- [46] Radford M. Neal (1993), “Probabilistic Inference Using Markov Chain Monte Carlo Methods,” technical report CRG-TR-93-1, Dept. of Computer Science, University of Toronto.
- [47] Jeffrey J. Neil, and G. Larry Bretthorst (1993), “On the Use of Bayesian Probability Theory for Analysis of Exponential Decay Data: An Example Taken from Intravoxel Incoherent Motion Experiments,” *Magn. Reson. in Med.*, **29**, pp. 642–647.
- [48] H. Nyquist (1924), “Certain Factors Affecting Telegraph Speed,” *Bell System Technical Journal*, **3**, pp. 324-346.
- [49] H. Nyquist (1928), “Certain Topics in Telegraph Transmission Theory,” *Transactions AIEE*, **3**, pp. 617-644.
- [50] William H. Press, Saul A. Teukolsky, William T. Vetterling and Brian P. Flannery (1992), “Numerical Recipes The Art of Scientific Computing Second Edition,” Cambridge University Press, Cambridge UK.
- [51] Emanuel Parzen (1962), “On Estimation of a Probability Density Function and Mode,” *Annals of Mathematical Statistics* **33**, 1065–1076
- [52] Karl Pearson (1895), “Contributions to the Mathematical Theory of Evolution. II. Skew Variation in Homogeneous Material,” *Phil. Trans. R. Soc. A* **186**, 343–326.

- [53] Murray Rosenblatt, "Remarks on Some Nonparametric Estimates of a Density Function," *Annals of Mathematical Statistics* **27**, 832–837 (1956).
- [54] Jeffery D. Scargle (1981), "Studies in Astronomical Time Series Analysis I. Random Process In The Time Domain," *Astrophysical Journal Supplement Series*, **45**, pp. 1-71.
- [55] Jeffery D. Scargle (1982), "Studies in Astronomical Time Series Analysis II. Statistical Aspects of Spectral Analysis of Unevenly Sampled Data," *Astrophysical Journal*, **263**, pp. 835-853.
- [56] Jeffery D. Scargle (1989), "Studies in Astronomical Time Series Analysis. III. Fourier Transforms, Autocorrelation Functions, and Cross-correlation Functions of Unevenly Spaced Data," *Astrophysical Journal*, **343**, pp. 874-887.
- [57] Arthur Schuster (1905), "The Periodogram and its Optical Analogy," *Proceedings of the Royal Society of London*, **77**, p. 136-140.
- [58] Claude E. Shannon (1948), "A Mathematical Theory of Communication," *Bell Syst. Tech. J.*, **27**, pp. 379-423.
- [59] John E. Shore, and Rodney W. Johnson (1981), "Properties of cross-entropy minimization," *IEEE Trans. on Information Theory*, **IT-27**, No. 4, pp. 472-482.
- [60] John E. Shore and Rodney W. Johnson (1980), "Axiomatic derivation of the principle of maximum entropy and the principle of minimum cross-entropy," *IEEE Trans. on Information Theory*, **IT-26** (1), pp. 26-37.
- [61] Devinderjit Sivia, and John Skilling (2006), "Data Analysis: A Bayesian Tutorial," Oxford University Press, USA.
- [62] Edward O. Stejskal and Tanner, J. E. (1965), "Spin Diffusion Measurements: Spin Echoes in the Presence of a Time-Dependent Field Gradient." *Journal of Chemical Physics*, **42** (1), pp. 288-292.
- [63] D. G. Taylor and Bushell, M. C. (1985), "The spatial mapping of translational diffusion coefficients by the NMR imaging technique," *Physics in Medicine and Biology*, **30** (4), pp. 345-349.
- [64] Myron Tribus (1969), "Rational Descriptions, Decisions and Designs," Pergamon Press, Oxford.
- [65] P. M. Woodward (1953), "Probability and Information Theory, with Applications to Radar," McGraw-Hill, N. Y. Second edition (1987); R. E. Krieger Pub. Co., Malabar, Florida.
- [66] Arnold Zellner (1971), "An Introduction to Bayesian Inference in Econometrics," John Wiley and Sons, New York.