

Bayesian Analysis Users Guide  
Release 4.00, Manual Version 1

G. Larry Bretthorst  
Biomedical MR Laboratory  
Washington University School Of Medicine,  
Campus Box 8227  
Room 2313, East Bldg.,  
4525 Scott Ave.  
St. Louis MO 63110  
<http://bayes.wustl.edu>  
Email: [larry@bayes.wustl.edu](mailto:larry@bayes.wustl.edu)

October 21, 2016



# Contents

<b>Manual Status</b>	<b>16</b>
<b>1 An Overview Of The Bayesian Analysis Software</b>	<b>19</b>
1.1 The Server Software	19
1.2 The Client Interface	22
1.2.1 The Global Pull Down Menus	24
1.2.2 The Package Interface	24
1.2.3 The Viewers	27
<b>2 Installing the Software</b>	<b>29</b>
<b>3 the Client Interface</b>	<b>33</b>
3.1 The Global Pull Down Menus	35
3.1.1 the Files menu	35
3.1.2 the Packages menu	40
3.1.3 the WorkDir menu	45
3.1.4 the Settings menu	46
3.1.5 the Utilities menu	50
3.1.6 the Help menu	50
3.2 The Submit Job To Server area	51
3.3 The Server area	52
3.4 Interface Viewers	52
3.4.1 the Ascii Data Viewer	53
3.4.2 the fid Data Viewer	53
3.4.3 Image Viewer	59
3.4.3.1 the Image List area	59
3.4.3.2 the Set Image area	62
3.4.3.3 the Image Viewing area	62
3.4.3.4 the Grayscale area on the bottom	63
3.4.3.5 the Pixel Info area	63
3.4.3.6 the Image Statistics area	64
3.4.4 Prior Viewer	65
3.4.5 Fid Model Viewer	68
3.4.5.1 The fid Model Format	70

3.4.5.2	The Fid Model Reports . . . . .	71
3.4.6	Plot Results Viewer . . . . .	71
3.4.7	Text Results Viewer . . . . .	74
3.4.8	Files Viewer . . . . .	80
3.5	Common Interface Plots . . . . .	80
3.5.1	Data, Model And Residual Plot . . . . .	81
3.5.2	Posterior Probability For A Parameter . . . . .	82
3.5.3	Maximum Entropy Histograms . . . . .	83
3.5.4	Markov Monte Carlo Samples . . . . .	83
3.5.5	Probability Vs Parameter Samples plot . . . . .	86
3.5.6	Expected Log Likelihood Plot . . . . .	88
3.5.7	Scatter Plots . . . . .	88
3.5.8	Logarithm of the Posterior Probability Plot . . . . .	91
3.5.9	Fortran/C Code Viewer . . . . .	91
3.5.9.1	Fortran/C Model Viewer Popup Editor . . . . .	94
<b>4</b>	<b>An Introduction to Bayesian Probability Theory</b>	<b>99</b>
4.1	The Rules of Probability Theory . . . . .	99
4.2	Assigning Probabilities . . . . .	102
4.3	Example: Parameter Estimation . . . . .	109
4.3.1	Define The Problem . . . . .	110
4.3.1.1	The Discrete Fourier Transform . . . . .	110
4.3.1.2	Aliases . . . . .	113
4.3.2	State The Model—Single-Frequency Estimation . . . . .	114
4.3.3	Apply Probability Theory . . . . .	115
4.3.4	Assign The Probabilities . . . . .	118
4.3.5	Evaluate The Sums and Integrals . . . . .	120
4.3.6	How Probability Generalizes The Discrete Fourier Transform . . . . .	123
4.3.7	Aliasing . . . . .	126
4.3.8	Parameter Estimates . . . . .	132
4.4	Summary and Conclusions . . . . .	136
<b>5</b>	<b>Given Exponential Model</b>	<b>137</b>
5.1	The Bayesian Calculation . . . . .	139
5.2	Outputs From The Given Exponential Package . . . . .	141
<b>6</b>	<b>Unknown Number of Exponentials</b>	<b>143</b>
6.1	The Bayesian Calculations . . . . .	145
6.2	Outputs From The Unknown Number of Exponentials Package . . . . .	148
<b>7</b>	<b>Inversion Recovery</b>	<b>151</b>
7.1	The Bayesian Calculation . . . . .	153
7.2	Outputs From The Inversion Recovery Package . . . . .	154

<b>8</b>	<b>Bayes Analyze</b>	<b>155</b>
8.1	Bayes Model	159
8.2	The Bayes Analyze Model Equation	161
8.3	The Bayesian Calculations	167
8.4	Levenberg-Marquardt And Newton-Raphson	171
8.5	Outputs From The Bayes Analyze Package	176
8.5.1	The “bayes.params.nnnn” Files	177
8.5.1.1	The Bayes Analyze File Header	178
8.5.1.2	The Global Parameters	182
8.5.1.3	The Model Components	184
8.5.2	The “bayes.model.nnnn” Files	185
8.5.3	The “bayes.output.nnnn” File	186
8.5.4	The “bayes.probabilities.nnnn” File	190
8.5.5	The “bayes.log.nnnn” File	193
8.5.6	The “bayes.status.nnnn” and “bayes.accepted.nnnn” Files	196
8.5.7	The “bayes.model.nnnn” File	197
8.5.8	The “bayes.summary1.nnnn” File	198
8.5.9	The “bayes.summary2.nnnn” File	199
8.5.10	The “bayes.summary3.nnnn” File	200
8.6	Bayes Analyze Error Messages	200
<b>9</b>	<b>Big Peak/Little Peak</b>	<b>207</b>
9.1	The Bayesian Calculation	209
9.2	Outputs From The Big Peak/Little Peak Package	216
<b>10</b>	<b>Metabolic Analysis</b>	<b>219</b>
10.1	The Metabolic Model	223
10.2	The Bayesian Calculation	225
10.3	The Metabolite Models	228
10.3.1	The IPGD_D2O Metabolite	228
10.3.2	The Glutamate.2.0 Metabolite	232
10.3.3	The Glutamate.3.0 Metabolite	235
10.4	The Example Metabolite	236
10.5	Outputs From The Bayes Metabolite Package	238
<b>11</b>	<b>Find Resonances</b>	<b>239</b>
11.1	The Bayesian Calculations	241
11.2	Outputs From The Bayes Find Resonances Package	246
<b>12</b>	<b>Diffusion Tensor Analysis</b>	<b>247</b>
12.1	The Bayesian Calculation	249
12.2	Using The Package	254
<b>13</b>	<b>Big Magnetization Transfer</b>	<b>259</b>
13.1	The Bayesian Calculation	259
13.2	Outputs From The Big Magnetization Transfer Package	262

<b>14 Magnetization Transfer</b>	<b>265</b>
14.1 The Bayesian Calculation	267
14.2 Using The Package	271
<b>15 Magnetization Transfer Kinetics</b>	<b>275</b>
15.1 The Bayesian Calculation	277
15.2 Using The Package	281
<b>16 Given Polynomial Order</b>	<b>285</b>
16.1 The Bayesian Calculation	287
16.1.1 Gram-Schmidt	287
16.1.2 The Bayesian Calculation	288
16.2 Outputs From the Given Polynomial Order Package	290
<b>17 Unknown Polynomial Order</b>	<b>293</b>
17.1 Bayesian Calculations	295
17.1.1 Assigning Priors	296
17.1.2 Assigning The Joint Posterior Probability	297
17.2 Outputs From the Unknown Polynomial Order Package	299
<b>18 Errors In Variables</b>	<b>303</b>
18.1 The Bayesian Calculation	305
18.2 Outputs From The Errors In Variables Package	308
<b>19 Behrens-Fisher</b>	<b>311</b>
19.1 Bayesian Calculation	311
19.1.1 The Four Model Selection Probabilities	314
19.1.1.1 The Means And Variances Are The Same	315
19.1.1.2 The Mean Are The Same And The Variances Differ	317
19.1.1.3 The Means Differ And The Variances Are The Same	318
19.1.1.4 The Means And Variances Differ	319
19.1.2 The Derived Probabilities	320
19.1.3 Parameter Estimation	321
19.2 Outputs From Behrens-Fisher Package	322
<b>20 Enter Ascii Model</b>	<b>329</b>
20.1 The Bayesian Calculation	331
20.1.1 The Bayesian Calculations Using Eq. (20.1)	331
20.1.2 The Bayesian Calculations Using Eq. (20.2)	332
20.2 Outputs Form The Enter Ascii Model Package	335
<b>21 Enter Ascii Model Selection</b>	<b>337</b>
21.1 The Bayesian Calculations	339
21.1.1 The Direct Probability With No Amplitude Marginalization	340
21.1.2 The Direct Probability With Amplitude Marginalization	342
21.1.2.1 Marginalizing the Amplitudes	343
21.1.2.2 Marginalizing The Noise Standard Deviation	348

21.2	Outputs Form The Enter Ascii Model Package . . . . .	349
<b>26</b>	<b>Phasing An Image</b>	<b>395</b>
26.1	The Bayesian Calculation . . . . .	396
26.2	Using The Package . . . . .	402
<b>27</b>	<b>Phasing An Image Using Non-Linear Phases</b>	<b>405</b>
27.1	The Model Equation . . . . .	405
27.2	The Bayesian Calculations . . . . .	407
27.3	The Interfaces To The Nonlinear Phasing Routine . . . . .	409
<b>28</b>	<b>Analyze Image Pixel</b>	<b>411</b>
28.1	Modification History . . . . .	413
<b>29</b>	<b>The Image Model Selection Package</b>	<b>415</b>
29.1	The Bayesian Calculations . . . . .	417
29.2	Outputs Form The Image Model Selection Package . . . . .	418
<b>A</b>	<b>Ascii Data File Formats</b>	<b>423</b>
A.1	Ascii Input Data Files . . . . .	423
A.2	Ascii Image File Formats . . . . .	424
A.3	The Abscissa File Format . . . . .	425
<b>B</b>	<b>Markov chain Monte Carlo With Simulated Annealing</b>	<b>439</b>
B.1	Metropolis-Hastings Algorithm . . . . .	440
B.2	Multiple Simulations . . . . .	441
B.3	Simulated Annealing . . . . .	442
B.4	The Annealing Schedule . . . . .	442
B.5	Killing Simulations . . . . .	443
B.6	the Proposal . . . . .	444
<b>C</b>	<b>Thermodynamic Integration</b>	<b>445</b>
<b>D</b>	<b>McMC Values Report</b>	<b>449</b>
<b>E</b>	<b>Writing Fortran/C Models</b>	<b>455</b>
E.1	Model Subroutines, No Marginalization . . . . .	455
E.2	The Parameter File . . . . .	458
E.3	The Subroutine Interface . . . . .	460
E.4	The Subroutine Declarations . . . . .	462
E.5	The Subroutine Body . . . . .	463
E.6	Model Subroutines With Marginalization . . . . .	464
<b>F</b>	<b>the Bayes Directory Organization</b>	<b>469</b>
<b>G</b>	<b>4dfp Overview</b>	<b>471</b>

**H Outlier Detection**

**Bibliography**



# List of Figures

1.1	The Start Up Window . . . . .	23
1.2	Example Package Exponential Interface . . . . .	25
2.1	Installation Kit For The Bayesian Analysis Software . . . . .	31
3.1	The Start Up Window . . . . .	34
3.2	The Files Menu . . . . .	35
3.3	The Files/Load Image Submenu . . . . .	37
3.4	The Packages Menu . . . . .	41
3.5	The Working Directory Menu . . . . .	46
3.6	The Working Directory Information Popup . . . . .	47
3.7	The Settings Pull Down Menu . . . . .	47
3.8	The McMC Parameters Popup . . . . .	48
3.9	The Edit Server Popup . . . . .	49
3.10	The Submit Job Widgets . . . . .	51
3.11	The Server Widgets Group . . . . .	52
3.12	The Ascii Data Viewer . . . . .	54
3.13	The Fid Data Viewer . . . . .	55
3.14	Fid Data Display Type . . . . .	56
3.15	Fid Data Options Menu . . . . .	58
3.16	The Image Viewer . . . . .	60
3.17	The Image Viewer Right Mouse Popup Menu . . . . .	61
3.18	The Prior Probability Viewer . . . . .	66
3.19	The Fid Model Viewer . . . . .	69
3.20	The Plot Results Viewer . . . . .	72
3.21	Plot Information Popup . . . . .	73
3.22	The Text Results Viewer . . . . .	75
3.23	The Bayes Condensed File . . . . .	78
3.24	Data, Model, And Resid Plot . . . . .	81
3.25	The Parameter Posterior Probabilities . . . . .	82
3.26	The Maximum Entropy Histograms . . . . .	84
3.27	The Parameter Samples Plot . . . . .	85
3.28	Posterior Probability Vs Parameter Value . . . . .	86
3.29	Posterior Probability Vs Parameter Value, A Skewed Example . . . . .	87
3.30	The Expected Value Of The Logarithm Of The Likelihood . . . . .	89

3.31	The Scatter Plots . . . . .	90
3.32	The Logarithm Of The Posterior Probability By Repeat Plot . . . . .	92
3.33	The Fortran/C Model Viewer . . . . .	93
3.34	The Fortran/C Code Editor . . . . .	95
4.1	Frequency Estimation Using The DFT . . . . .	112
4.2	Aliases . . . . .	113
4.3	Nonuniformly Nonsimultaneously Sampled Sinusoid . . . . .	127
4.4	Alias Spacing . . . . .	128
4.5	Which Is The Critical Time . . . . .	130
4.6	Example, Frequency Estimation . . . . .	131
4.7	Estimating The Sinusoids Parameters . . . . .	133
5.1	The Given And Unknown Number Of Exponential Package Interface . . . . .	138
6.1	The Unknown Exponential Interface . . . . .	144
6.2	The Distribution Of Models . . . . .	149
6.3	The Posterior Probability For Exponential Model . . . . .	150
7.1	The Inversion Recovery Interface . . . . .	152
8.1	Bayes Analyze Interface . . . . .	156
8.2	Bayes Analyze Fid Model Viewer . . . . .	160
8.3	The Bayes Analyze File Header . . . . .	179
8.4	The bayes.noise File . . . . .	180
8.5	Bayes Analyze Global Parameters . . . . .	183
8.6	The Third Section Of The Parameter File . . . . .	184
8.7	Example Of An Initial Model In The Output File . . . . .	187
8.8	Base 10 Logarithm Of The Odds . . . . .	187
8.9	A Small Sample Of The Output Report . . . . .	188
8.10	Bayes Analyze Uncorrelated Output . . . . .	189
8.11	The bayes.proBABILITIES.nnnn File . . . . .	191
8.12	The bayes.log.nnnn File . . . . .	193
8.13	The bayes.status.nnnn File . . . . .	196
8.14	The bayes.model.nnnn File . . . . .	197
8.15	The bayes.model.nnnn File Uncorrelated Resonances . . . . .	198
8.16	Bayes Analyze Summary Header . . . . .	198
8.17	The Summary2 (Best Summary) . . . . .	199
8.18	The Summary3 Report . . . . .	201
9.1	The Big Peak/Little Peak Interface . . . . .	208
9.2	The Time Dependent Parameters . . . . .	218
10.1	The Bayes Metabolite Interface . . . . .	220
10.2	The Bayes Metabolite Viewer . . . . .	222
10.3	Bayes Metabolite Parameters And Probabilities List . . . . .	227
10.4	The IPGD_D20 Metabolite . . . . .	229

10.5	Bayes Metabolite IPGD_D20 Spectrum . . . . .	230
10.6	Bayes Metabolite, The Fraction of Glucose . . . . .	231
10.7	Glutamate Example Spectrum . . . . .	233
10.8	Estimating The $F_{c0}$ , $y$ and $F_{a0}$ Parameters . . . . .	236
10.9	Bayes Metabolite, The Ethyl Ether Example . . . . .	237
11.1	The Find Resonances Interface With The Ethyl Ether Spectrum . . . . .	240
12.1	The Diffusion Tensor Package Interface . . . . .	248
12.2	Diffusion Tensor Parameter Estimates . . . . .	256
12.3	Diffusion Tensor Posterior Probability For The Model . . . . .	257
13.1	The Big Magnetization Package Interface . . . . .	260
13.2	Big Magnetization Transfer Example Fid . . . . .	263
13.3	Big Magnetization Transfer Expansion . . . . .	263
13.4	Big Magnetization Transfer Peak Pick . . . . .	264
14.1	The Magnetization Transfer Package Interface . . . . .	266
14.2	Magnetization Transfer Package Peak Picking . . . . .	272
14.3	Magnetization Transfer Example Data . . . . .	273
14.4	Magnetization Transfer Example Spectrum . . . . .	274
15.1	Magnetization Transfer Kinetics Package Interface . . . . .	276
15.2	Magnetization Transfer Kinetics Package Arrhenius Plot . . . . .	282
15.3	Magnetization Transfer Kinetics Water Viscosity Table . . . . .	283
16.1	Given Polynomial Order Package Interface . . . . .	286
16.2	Given Polynomial Order Scatter Plot . . . . .	291
17.1	Unknown Polynomial Order Package Interface . . . . .	294
17.2	The Distribution of Models On The Console Log . . . . .	298
17.3	The Posterior Probability For The Polynomial Order . . . . .	300
18.1	The Errors In Variables Package Interface . . . . .	304
18.2	The McMC Values File Produced By The Errors In Variables Package . . . . .	310
19.1	The Behrens-Fisher Interface . . . . .	312
19.2	Behrens-Fisher Hypotheses Tested . . . . .	313
19.3	Behrens-Fisher Console Log . . . . .	323
19.4	Behrens-Fisher Status Listing . . . . .	324
19.5	Behrens-Fisher McMC Values File, The Preamble . . . . .	325
19.6	Behrens-Fisher McMC Values File, The Middle . . . . .	326
19.7	Behrens-Fisher McMC Values File, The End . . . . .	327
20.1	Enter Ascii Model Package Interface . . . . .	330
21.1	The Enter Ascii Model Selection Package Interface . . . . .	338

26.1	Absorption Model Images . . . . .	396
26.2	The Interface To The Image Phasing Package . . . . .	397
26.3	Linear Phasing Package The Console Log . . . . .	403
27.1	Nonlinear Phasing Example . . . . .	406
27.2	The Interface To The Nonlinear Phasing Package . . . . .	410
28.1	The Interface To The Analyze Image Pixels Package . . . . .	412
29.1	The Interface To The Image Model Selection Package . . . . .	416
29.2	Single Exponential Example Image . . . . .	419
29.3	Single Exponential Example Data . . . . .	420
29.4	Posterior Probability For The ExpOneNoConst Model . . . . .	421
A.1	Ascii Data File Format . . . . .	424
D.1	The McMC Values Report Header . . . . .	450
D.2	McMC Values Report, The Middle . . . . .	451
D.3	The McMC Values Report, The End . . . . .	452
E.1	Writing Models A Fortran Example . . . . .	456
E.2	Writing Models A C Example . . . . .	457
E.3	Writing Models, The Parameter File . . . . .	459
E.4	Writing Models Fortran Declarations . . . . .	463
E.5	Writing Models Fortran Example . . . . .	466
E.6	Writing Models The Parameter File . . . . .	467
G.1	Example FDF File Header . . . . .	473
H.1	The Posterior Probability For The Number of Outliers . . . . .	476
H.2	The Data, Model and Residual Plot With Outliers . . . . .	478

# List of Tables

8.1	Multiplet Relative Amplitudes . . . . .	165
8.2	Bayes Analyze Models . . . . .	181
8.3	Bayes Analyze Short Descriptions . . . . .	195

## Chapter 14

# Magnetization Transfer

The Magnetization transfer package analyzes two site magnetization exchange data. This data is obtained from a peak pick, a Bayes Analyze file or it can be manually entered and loaded as an Ascii file. The Ascii data used in this package are generated from an Fid. The Fid data should be an arrayed inversion recovery data set. If we call the two sites that are exchanging magnetization Site A, and Site B, then the data should be in inversion recover data set where Site A was inverted. You should also have a data set where Site B was inverted. Finally, you can also invert both sites and that data may also be used. Preferably, you should array these Fid so as to obtain as many data values as possible on each recovery curve, the more data you have the more precise you parameter estimates will be. Note, that while it is not recommended, a single inversion recover data set can be used and you will be able to get exchange rates. However, because of the limited about of data, they will probably be highly uncertain. The interface to this package is shown in Fig. 14.1. To use this package, you must do the following:

**Select** the Magnetization Transfer package from the Package menu.

**Load** at least one and preferably two three column Ascii data sets using the Files menu. The three columns are the abscissa value, and the amplitude or peak value of the Site A and B magnetization. Additionally, you can load an arrayed Fid and then use a double cursor to mark the center of the two exchanging peaks and use the “Get Peak” button on the bottom right of the Fid viewer. When a data set has been successfully loaded a plot contain the two sites is displayed in the Ascii Data viewer. For Fids, to load a second peak pick or Bayes Analyze file, simply load a second Fid and either load a peak pick or a Bayes Analyze file. Finally, if you have analyzed this Fid using Bayes Analyze you can load the resonance amplitude from the Bayes Analyze files using the “Files/Load Bayes Analyze” menu.

**Check** the Analysis Options/Find Outliers box if you suspect outliers are present in the data.

**Normally** the prior probabilities for the parameters would normally have to be reviewed here. However, the calculations are done using a variable transformation, sum and difference variables and this change of variables makes determining prior ranges so easy that the package does it automatically.

**Select** the server that is to process the analysis.

Figure 14.1: The Magnetization Transfer Package Interface

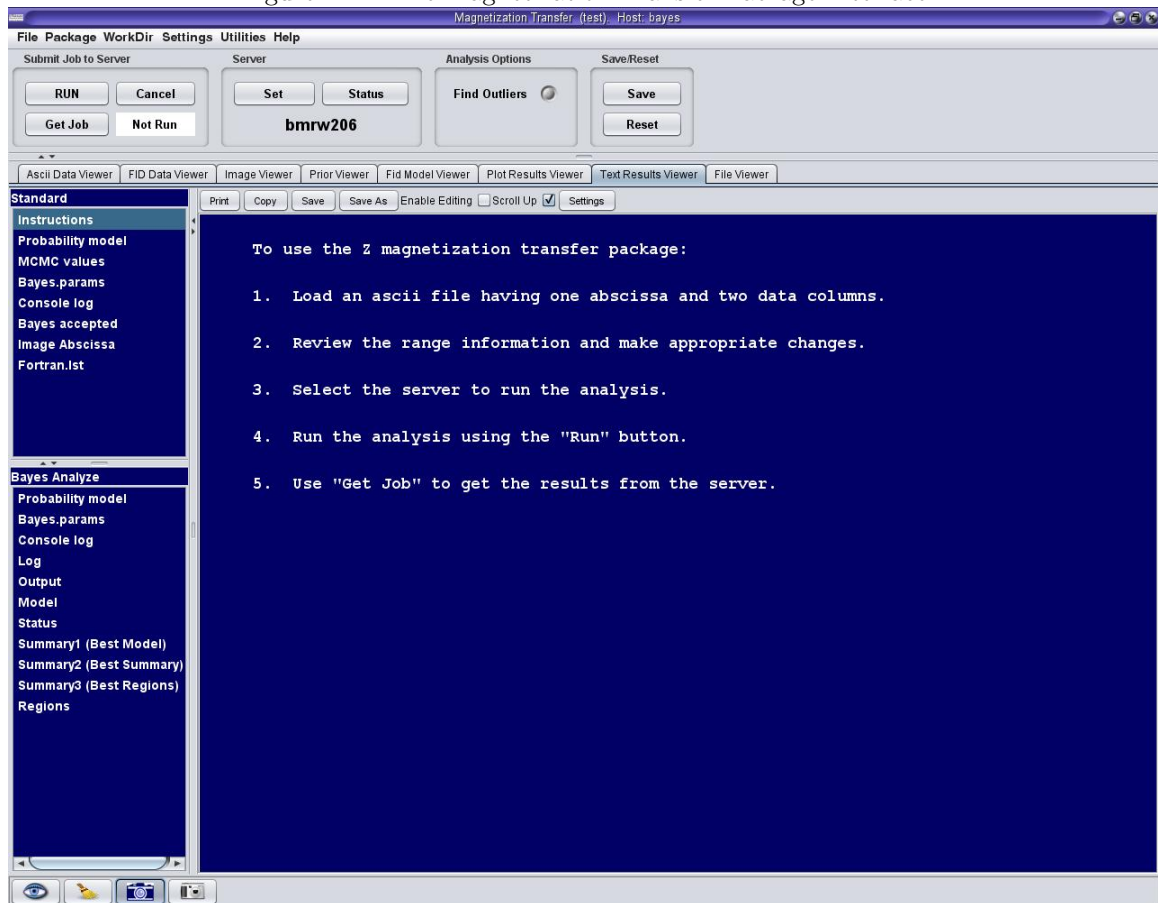


Figure 14.1: The magnetization transfer packages analyzes one or more data set using the equations governing two site magnetization exchange. The inferred parameters are the two exchange rates and the two relaxation rates. For more on the actual calculations and the widgets see the text.

**Check** the status of the selected server to determine if the server is busy, change to another server if the selected server is busy.

**Run** the the analysis on the selected server by activating the Run button.

**Get** the the results of the analysis by activating the Get Job button. If the analysis is running, this button will return the Accepted report containing the status of the current run. Otherwise, it will fetch and display the results from the current analysis.

## 14.1 The Bayesian Calculation

The two site magnetization transfer package solves problems involving magnetization transfer. If we designate the two sites as “*a*” and “*b*” respectively then the magnetization transfer model of the “*a*” site is related to the data by

$$d_a(t_i) = M_a(t_i) + \text{error} \quad (14.1)$$

and similarly for the “*b*” site,

$$d_b(t_i) = M_b(t_i) + \text{error} \quad (14.2)$$

where  $M_a(t)$  and  $M_b(t)$  are the solution to the Block-McConnell equations, and “error” represents noise in the data and should not be taken to mean that this noise is the same in both data sets. McConnell’s modification to the Block equation is given by

$$\frac{dM_a(t)}{dt} = -R_{1a}[M_a(t) - M_a(\infty)] - K_{ab}M_a(t) + K_{ba}M_b(t) \quad (14.3)$$

$$\frac{dM_b(t)}{dt} = -R_{1b}[M_b(t) - M_b(\infty)] - K_{ba}M_b(t) + K_{ab}M_a(t) \quad (14.4)$$

where  $R_{1a}$  and  $R_{1b}$  are the relaxation rates for the “*a*” and “*b*” sites,  $K_{ba}$  is the rate at which magnetization goes from the “*b*” to the “*a*” site. Similarly  $K_{ab}$  is the rate at which magnetization exchanges from the “*a*” to the “*b*” site.

These equations are coupled linear first order differential equations and their bi-exponential solution is given by

$$M_a(t) = H(t)M_a(0) + G(t)M_b(0) + \left[1 - H(t) - G(t)\frac{K_{ab}}{K_{ba}}\right] M_a(\infty) \quad (14.5)$$

$$M_b(t) = I(t)M_b(0) - J(t)M_a(0) + \left[J(t) + [1 - I(t)]\frac{K_{ba}}{K_{ba}}\right] M_a(\infty) \quad (14.6)$$

where  $M_a(\infty)$  is the equilibrium magnetization for the “*a*” site, and  $M_a(0)$  and  $M_b(0)$  are the initial magnetization for the “*a*” and “*b*” sites. The functions  $G(t)$ ,  $H(t)$ ,  $I(t)$  and  $J(t)$  are defined as

$$G(t) = \frac{\exp\{\alpha_1 t\} - \exp\{\alpha_2 t\}}{U - V}, \quad (14.7)$$

$$H(t) = \frac{U \exp\{\alpha_2 t\} - V \exp\{\alpha_1 t\}}{U - V}, \quad (14.8)$$

$$I(t) = \frac{U \exp\{\alpha_1 t\} - V \exp\{\alpha_2 t\}}{U - V}, \quad (14.9)$$



$$J(t) = UVG(t), \quad (14.10)$$

with

$$U = \frac{\alpha_1 + K_{ab} + R_{1a}}{K_{ba}}, \quad (14.11)$$

$$V = \frac{\alpha_2 + K_{ab} + R_{1a}}{K_{ba}}. \quad (14.12)$$

The observed decay rates  $\alpha_1$  and  $\alpha_2$  are given by

$$\alpha_{1,2} = -\frac{R_{1a} + R_{1b} + K_{ab} + K_{ba}}{2} \pm \frac{1}{2} \sqrt{(R_{1a} - R_{1b} - K_{ab} - K_{ba})^2 + 4K_{ab}K_{ba}} \quad (14.13)$$

where  $\alpha_1$  takes the plus and  $\alpha_2$  the minus.

The four parameters of interest are the two exchange rates  $K_{ab}$  and  $K_{ba}$  and the two relaxation rates  $R_{1a}$  and  $R_{1b}$ . In addition there are three parameters per “a” and “b” site that must be included, two initial conditions magnetizations,  $M_a(0)$  and  $M_b(0)$  and one equilibrium magnetization  $M_a(\infty)$ . The equilibrium condition,

$$M_b(\infty) = M_a(\infty) \frac{K_{ab}}{K_{ba}} \quad (14.14)$$

was used to eliminate  $M_b(\infty)$ .

Before we start the process of computing the posterior probability for the parameters of interest,  $K_{ab}$ ,  $K_{ba}$ ,  $R_{1a}$  and  $R_{1b}$ , we are going to rewrite the model equations and the data into a form that is more convenient for the upcoming analytic calculation. First we are going to define a single data set  $D$  with data item  $d(t_i)$  that is the “a” site data followed by the “b” site data:

$$d(t_i) = \begin{cases} d_a(t_i) & \text{if } 1 \leq i \leq N \\ d_b(t_j) & \text{if } 1 \leq j \leq N \text{ with } j \equiv i - N \end{cases} \quad (14.15)$$

where the index  $i$  ranges from 1 up to  $2N$ . This definition is roughly like defining a complex data set with the “a” site being the real data and the “b” site being the imaginary data. Next we will rewrite Eqs. (14.1 and 14.2) into a single equation:

$$d(t_i) = \sum_{k=1}^3 M_k \mathcal{H}_k(t_i) + \text{error} \quad (14.16)$$

where  $M_1 \equiv M_a(0)$ ,  $M_2 \equiv M_b(0)$  and  $M_3 \equiv M_a(\infty)$ . Finally, the three functions  $\mathcal{H}_1(t_i)$ ,  $\mathcal{H}_2(t_i)$  and  $\mathcal{H}_3(t_i)$  are defined as

$$\mathcal{H}_1(t_i) = \begin{cases} H(t_i) & \text{if } 1 \leq i \leq N \\ -J(t_j) & \text{if } 1 \leq j \leq N \text{ with } j \equiv i - N \end{cases}, \quad (14.17)$$

$$\mathcal{H}_2(t_i) = \begin{cases} G(t_i) & \text{if } 1 \leq i \leq N \\ I(t_j) & \text{if } 1 \leq j \leq N \text{ with } j \equiv i - N \end{cases} \quad (14.18)$$

and

$$\mathcal{H}_3(t_i) = \begin{cases} (1 - H(t_i) - G(t_i)K_{ab}/K_{ba}) & \text{if } 1 \leq i \leq N \\ (J(t_j) + [1 - I(t_j)]K_{ab}/K_{ba}) & \text{if } 1 \leq j \leq N \text{ with } j \equiv i - N \end{cases}. \quad (14.19)$$

In magnetization transfer problems it is often possible to take multiple independent measurements of the exchanging spins. For example one can invert the “a” site and then watch the effect of the relaxation on the “b” site spins. Similarly, one could invert the “b” site and then even invert both sites simultaneously. Consequently, we are going to adopt the notation,  $d_j(t_i)$ , to designate the  $i$ th data item of the  $j$ th inversion recovery. Similarly,  $M_{j1}$  will be the initial “a” site magnetization for the  $j$ th inversion recovery. In what follows, it should be understood that different data sets may have different acquisition times, even though we will not adopt a notation to represent this.

The joint posterior probability for the four parameters of interest is represented symbolically by  $P(K_{ab}K_{ba}R_{1a}R_{1b}|DI)$ , where  $D$  are the amplitudes or peak intensities for the “a” and “b” sites for all of the data sets. If we designate the posterior probability for the parameters computed from the  $j$ th data set as  $P(K_{ab}K_{ba}R_{1a}R_{1b}|D_jI)$  then the posterior probability computed from all of the data is given by

$$P(K_{ab}K_{ba}R_{1a}R_{1b}|DI) = \prod_{j=1}^m P(K_{ab}K_{ba}R_{1a}R_{1b}|D_jI). \quad (14.20)$$

The right-hand side of this equation is computed from the joint posterior probability for all of the parameters:

$$P(K_{ab}K_{ba}R_{1a}R_{1b}|D_jI) = \int P(K_{ab}K_{ba}R_{1a}R_{1b}M_{j1}M_{j2}M_{j3}\sigma|D_jI)dM_{j1}dM_{j2}dM_{j3}d\sigma \quad (14.21)$$

where the amplitudes and standard deviation for the noise prior probability have been removed by marginalization. To compute the joint posterior probability for all of the parameters, we first factor the integrand using Bayes’ theorem:

$$\begin{aligned} P(K_{ab}K_{ba}R_{1a}R_{1b}M_{j1}M_{j2}M_{j3}\sigma|D_jI) &\propto P(K_{ab}K_{ba}R_{1a}R_{1b}M_{j1}M_{j2}M_{j3}\sigma_j|I) \\ &\times P(D_j|K_{ab}K_{ba}R_{1a}R_{1b}M_{j1}M_{j2}M_{j3}\sigma_jI). \end{aligned} \quad (14.22)$$

Next the joint prior probability for the parameters,  $P(K_{ab}K_{ba}R_{1a}R_{1b}M_{j1}M_{j2}M_{j3}\sigma_j|I)$ , is factored into a series of independent priors for each of the parameters:

$$\begin{aligned} P(K_{ab}K_{ba}R_{1a}R_{1b}M_{j1}M_{j2}M_{j3}\sigma_j|I) &= P(K_{ab}|I)P(K_{ba}|I)P(R_{1a}|I)P(R_{1b}|I) \\ &\times P(M_{j1}|I)P(M_{j2}|I)P(M_{j3}|I)P(\sigma_j|I). \end{aligned} \quad (14.23)$$

Using Eqs. (14.23,14.22,14.21 and 14.20), one obtains,

$$\begin{aligned} P(K_{ab}K_{ba}R_{1a}R_{1b}|DI) &= P(K_{ab}|I)P(K_{ba}|I)P(R_{1a}|I)P(R_{1b}|I) \\ &\times \prod_{j=1}^m \left[ \int dM_{j1}dM_{j2}dM_{j3}d\sigma_j P(\sigma_j|I) \right. \\ &\quad \times P(M_{j1}|I)P(M_{j2}|I)P(M_{j3}|I) \\ &\quad \left. \times P(D_j|K_{ab}K_{ba}R_{1a}R_{1b}M_{j1}M_{j2}M_{j3}\sigma_jI) \right] \end{aligned} \quad (14.24)$$

as the posterior probability for the parameters of interest.

We have now reached the point in the calculation where we must assign numerical values to represent these probabilities. The prior probabilities for the four parameters of interest are assigned

as Gaussians that are constructed out of the Low-High ranges that are input on the interface. If  $x$  denotes one of the four parameters of interest, then its prior probability is given by

$$P(x|I) \propto \begin{cases} \exp \left\{ -\frac{(\text{Mean}-x)^2}{2\text{Sd}^2} \right\} & \text{if } \text{Low} \leq x \leq \text{High} \\ 0 & \text{otherwise} \end{cases} \quad (14.25)$$

where ‘‘Low’’ and ‘‘High’’ appropriate inputs from the interface, ‘‘Mean’’ is the average of the low and high, and ‘‘Sd’’ is set so that the High-Low interval represents a 3 standard deviation interval.

The prior probabilities for the standard deviation of the noise was assigned as a Jeffreys’ prior,  $1/\sigma_j$ . The prior probability for the amplitudes was assigned using a uniform prior probability. Finally, the likelihoods were assigned using a Gaussian of standard deviation  $\sigma_j$ , one obtains

$$\begin{aligned} P(K_{ab}K_{ba}R_{1a}R_{1b}|DI) &\propto P(K_{ab}|I)P(K_{ba}|I)P(R_{1a}|I)P(R_{1b}|I) \\ &\times \prod_{j=1}^m \left[ \int dM_{j1}dM_{j2}dM_{j3} \frac{d\sigma_j}{\sigma_j} (2\pi\sigma_j^2)^{-N} \exp \left\{ -\frac{Q_j}{2\sigma_j^2} \right\} \right] \end{aligned} \quad (14.26)$$

where we have left the four prior probabilities for the parameters of interest in their symbolic form, and

$$\begin{aligned} Q_j &\equiv \sum_{i=1}^{2N_j} \left[ d_j(t_i) - \sum_{\ell=1}^3 M_{j\ell} \mathcal{H}_j(t_i) \right]^2 \\ &= 2N_j(\overline{d^2})_j - 2 \sum_{\ell=1}^3 M_{j\ell} T_{j\ell} + \sum_{k=1}^3 \sum_{l=1}^3 (g_{kl})_j M_{jk} M_{jl}. \end{aligned} \quad (14.27)$$

The mean-squared data value,  $(\overline{d^2})_j$ , for the  $j$ th inversion is defined as

$$(\overline{d^2})_j = \frac{1}{2N_j} \sum_{i=1}^{2N_j} d_j(t_i)^2 \equiv \frac{1}{2N_j} \sum_{i=1}^{N_j} d_{aj}(t_i)^2 + d_{bj}(t_i)^2 \quad (14.28)$$

where  $N_j$  is the number of complex data values in the  $j$ th inversion. The projection of the  $\ell$ th model function onto the  $j$ th inversion,  $T_{j\ell}$ , is given by

$$T_{j\ell} = \sum_{i=1}^{2N_j} d_j(t_i) \mathcal{H}_{j\ell}(t_i). \quad (14.29)$$

The matrix  $(g_{kl})_j$  is defined by

$$(g_{kl})_j \equiv \sum_{i=1}^{2N_j} \mathcal{H}_{jk}(t_i) \mathcal{H}_{j\ell}(t_i). \quad (14.30)$$

This matrix will depend on the individual inversion if the delay times differ from one inversion to another.

Evaluating the integrals over the amplitudes and standard deviations is straightforward, if not tedious. Evaluating the amplitude integrals one obtains:

$$\begin{aligned} P(K_{ab}K_{ba}R_{1a}R_{1b}|DI) &\propto P(K_{ab}|I)P(K_{ba}|I)P(R_{1a}|I)P(R_{1b}|I) \\ &\times \prod_{j=1}^m \int \frac{d\sigma_j}{\sigma_j} |g_{kl}|_j^{-\frac{1}{2}} (2\pi\sigma_j^2)^{-N+3} \exp \left\{ -\frac{2N_j(\overline{d^2})_j - (\overline{h^2})_j}{2\sigma_j^2} \right\} \end{aligned} \quad (14.31)$$

where  $|g_{kl}|_j$  is the determinant of the  $(g_{jk})_j$  matrix. The sufficient statistic,  $(\overline{h^2})_j$ , is the total-squared projection of the model onto the data for the  $j$ th inversion:

$$(\overline{h^2})_j = \sum_{k=1}^3 T_{kj} \hat{M}_{kj}. \quad (14.32)$$

The  $\hat{M}_{kj}$ , are given by the solution to the following linear set of equations:

$$\sum_{k=1}^3 (g_{lk})_j \hat{M}_{kj} = T_{lj} \quad (1 \leq l \leq 3). \quad (14.33)$$

The integrals over the standard deviations of the noise prior probabilities may all be transformed into Gamma functions and evaluating such integrals is straightforward, one obtains

$$\begin{aligned} P(K_{ab}K_{ba}R_{1a}R_{1b}|D_aD_bI) &\propto P(K_{ab}|I)P(K_{ba}|I)P(R_{1a}|I)P(R_{1b}|I) \\ &\times \prod_{j=1}^m |g_{kl}|_j^{-\frac{1}{2}} \left[ 2N_j(\overline{d^2})_j - (\overline{h^2})_j \right]^{\frac{3-N_j}{2}} \end{aligned} \quad (14.34)$$

where we have dropped a number of constants that cancel when this distribution is normalized.

Markov chain Monte Carlo with simulated annealing is used to draw samples from the joint posterior probability for  $K_{ab}$ ,  $K_{ba}$ ,  $R_{1a}$  and  $R_{1b}$ , Eq. (14.34). These samples are used to approximate the marginal posterior probability for each of the parameters separately. In addition, the program also uses these samples to form an approximation to the posterior probability for the volume fractions,

$$p_a = \frac{K_{ba}}{K_{ab} + K_{ba}} \quad \text{and} \quad p_b = \frac{K_{ab}}{K_{ab} + K_{ba}}, \quad (14.35)$$

the exchange times,

$$\tau_{ab} = \frac{1}{K_{ab}} \quad \text{and} \quad \tau_{ba} = \frac{1}{K_{ba}}, \quad (14.36)$$

and the relaxation times,

$$T_{1a} = \frac{1}{R_{1a}} \quad \text{and} \quad T_{1b} = \frac{1}{R_{1b}}. \quad (14.37)$$

Note that in each of these calculations involves a change of variables of the form  $Y = 1/X$ . This change of variables introduces a factor of the form  $dY = -dX X^{-2}$  into the posterior. This factor shift the location of the maximum posterior probability between the paired variables. Consequently, the maximum posterior probability estimate for  $Y$  is not in general equal to  $1/X$ .

## 14.2 Using The Package

The Text outputs files from the Magnetization Transfer packages consist of: “Bayes.prob.model,” “MtZ.mcmc.values,” “Bayes.params,” “Console.log,” “Bayes.accepted” and a “Bayes.Condensed.File.” These output files can be viewed using the Text Viewer or they can be viewed using File Viewer by navigating to the current working directory and then selecting the files. The format of the

Figure 14.2: Magnetization Transfer Package Peak Picking

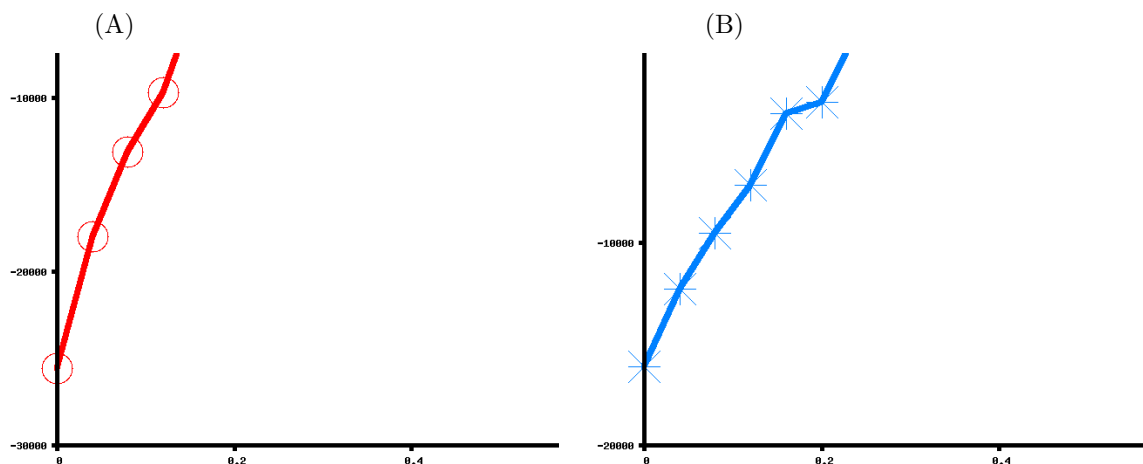


Figure 14.2: The peak intensities of the exchanging sites are analyzed by the magnetization exchange package, consequently, as illustrated here, the data are three column Ascii: a time, and two peak intensities. These peak intensities may be loaded from a peak pick, a Bayes Analyze run, or from an Ascii file. The example shown in this figure is from an peak pick. Panel (A) is the peak intensities when the left-hand peak was inverted and Panel (B) is the peak intensities when the right-hand peak was inverted. The data shown in Panel (A) was generated using a peak pick of the inversion recover spectrum shown in Figure 14.3.

mcmc.values report is discussed in Appendix D and the other reports are discussed in Chapter 3. Additionally, the “Plot Results Viewer” can be used to view the output probability density functions. In addition to the standard data, model and residual plots there are probability density functions for the decay rate constants, decay times, the amplitudes for each data set for each exponential and finally there are probability density functions for the standard deviation of the noise in each data set.

The data used by this package can be multiple three column Ascii files, see Fig. 14.2 for an example of this data. The data consists of a time axis and the left and right-hand peak intensity from an inversion recover experiment. The data can be loaded in one of three ways: First, a three column Ascii file can be directly loaded using the Files menu; Second, The spectrum of a magnetization transfer inversion recovery Fid can be loaded and the peak amplitudes can be extracted and loaded. Third, the “File/Load Ascii/Bayes Analyze File” button can be used to extract the peak amplitudes from the currently loaded Bayes Analyze files. If Bayes Analyze has not been run on this Fid, you must run it before you can use this option.

Figure 14.3 is the first trace in the spectrum of the inversion recover used to generate the data shown in Fig. 14.2(A). Figure 14.4 is a plot of the spectrum of the two exchanging resonances for each delay time in the inversion recover experiment shown in Fig. 14.3. Note how the left-hand resonances starts inverted and as a function of the delay time it rather quickly recovers. If you examine Fig. 14.4, which is a VnmrJ dssh display of the region around the two exchanging peaks, you can easily see that, initially both peaks have reduced amplitude and as the left-hand peak recovers, both peaks

Figure 14.3: Magnetization Transfer Example Data

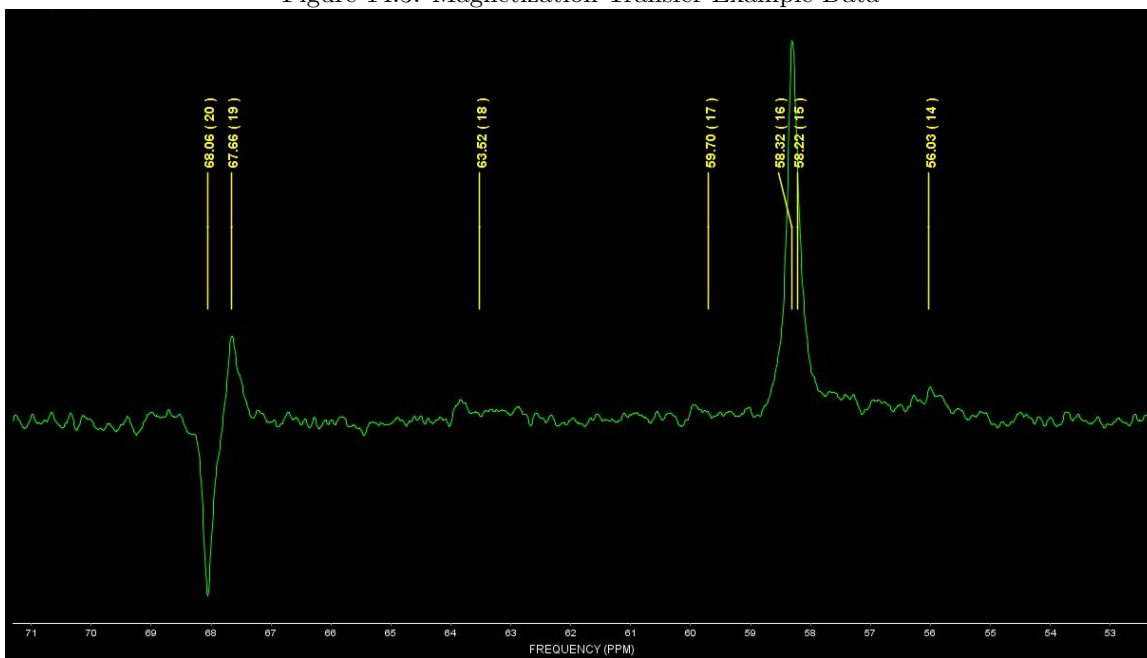


Figure 14.3: The three columns are the abscissa, in this case a delay time, and the amplitudes of the resonances of the A and B Sites. In this spectrum Site A is the resonance near 68PPM, and Site B is near 67.66PPM. To load this data, place a double cursor on each resonance and hit the “Get Peak” button in the lower right. Alternately, If Bayes Analyze has been run on this data, the “File/Load Ascii/Bayes Analyze File” menu can be used to load the resonance amplitudes from the Bayes Analyze files.

Figure 14.4: Magnetization Transfer Example Spectrum

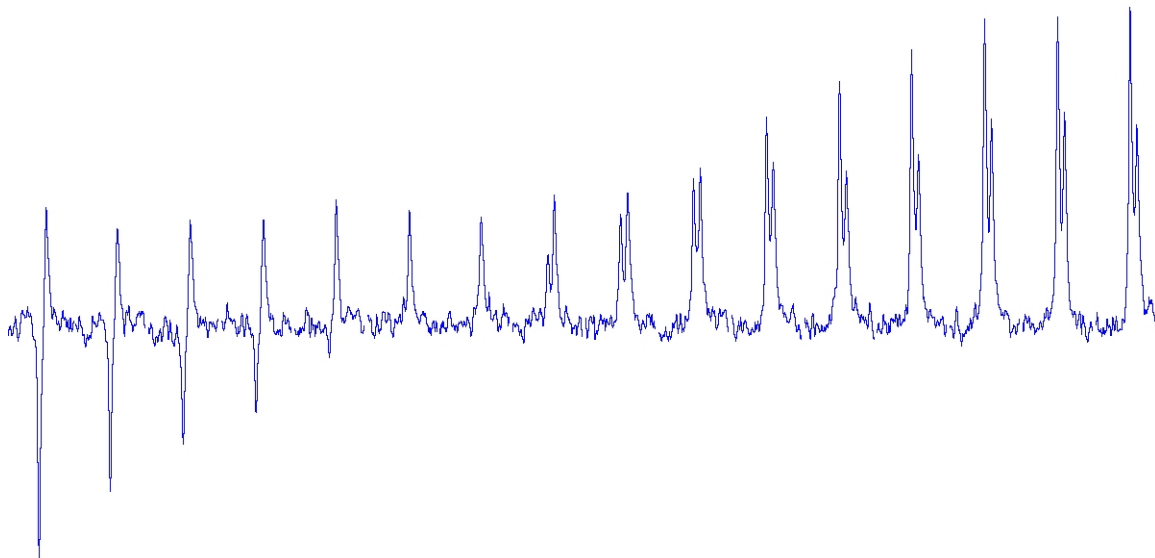


Figure 14.4: These spectra are a typical example of magnetization transfer data. Here the left-hand peak was selectively inverted and allowed to recover. The data analyzed by the magnetization transfer package is the peak intensity or amplitude of the two sites exchanging magnetization. These peak intensities may be loaded using the File menu, extracted using a double cursor and the Get Peak button, or Bayes Analyze can be used to estimate the amplitudes, and these amplitudes can be loaded using the File/Ascii File/Bayes Analyze menu.

increase in intensity. In the three column Ascii data shown in Fig 14.2(A) the left-hand peak shown in Fig 14.3 was selectively inverted and then allowed to relax back to equilibrium. It is the peak intensities of the two resonances exchanging magnetization that are analyzed by the magnetization package.

# Bibliography

- [1] Rev. Thomas Bayes (1763), “An Essay Toward Solving a Problem in the Doctrine of Chances,” *Philos. Trans. R. Soc. London*, **53**, pp. 370-418; reprinted in *Biometrika*, **45**, pp. 293-315 (1958), and *Facsimiles of Two Papers by Bayes*, with commentary by W. Edwards Deming, New York, Hafner, 1963.
- [2] G. Larry Bretthorst (1988), “Bayesian Spectrum Analysis and Parameter Estimation,” in *Lecture Notes in Statistics*, **48**, J. Berger, S. Fienberg, J. Gani, K. Krickenberg, and B. Singer (eds), Springer-Verlag, New York, New York.
- [3] G. Larry Bretthorst (1990), “An Introduction to Parameter Estimation Using Bayesian Probability Theory,” in *Maximum Entropy and Bayesian Methods*, Dartmouth College 1989, P. Fougère ed., pp. 53-79, Kluwer Academic Publishers, Dordrecht the Netherlands.
- [4] G. Larry Bretthorst (1990), “Bayesian Analysis I. Parameter Estimation Using Quadrature NMR Models” *J. Magn. Reson.*, **88**, pp. 533-551.
- [5] G. Larry Bretthorst (1990), “Bayesian Analysis II. Signal Detection And Model Selection” *J. Magn. Reson.*, **88**, pp. 552-570.
- [6] G. Larry Bretthorst (1990), “Bayesian Analysis III. Examples Relevant to NMR” *J. Magn. Reson.*, **88**, pp. 571-595.
- [7] G. Larry Bretthorst (1991), “Bayesian Analysis. IV. Noise and Computing Time Considerations,” *J. Magn. Reson.*, **93**, pp. 369-394.
- [8] G. Larry Bretthorst (1992), “Bayesian Analysis. V. Amplitude Estimation for Multiple Well-Separated Sinusoids,” *J. Magn. Reson.*, **98**, pp. 501-523.
- [9] G. Larry Bretthorst (1992), “Estimating The Ratio Of Two Amplitudes In Nuclear Magnetic Resonance Data,” in *Maximum Entropy and Bayesian Methods*, C. R. Smith et al. (eds.), pp. 67-77, Kluwer Academic Publishers, the Netherlands.
- [10] G. Larry Bretthorst (1993), “On The Difference In Means,” in *Physics & Probability Essays in honor of Edwin T. Jaynes*, W. T. Grandy and P. W. Milonni (eds.), pp. 177-194, Cambridge University Press, England.
- [11] G. Larry Bretthorst (1996), “An Introduction To Model Selection Using Bayesian Probability Theory,” in *Maximum Entropy and Bayesian Methods*, G. R. Heidbreder, ed., pp. 1-42, Kluwer Academic Publishers, Printed in the Netherlands.



- [12] G. Larry Bretthorst (1999), “The Near-Irrelevance of Sampling Frequency Distributions,” in *Maximum Entropy and Bayesian Methods*, W. von der Linden *et al.* (eds.), pp. 21-46, Kluwer Academic Publishers, the Netherlands.
- [13] G. Larry Bretthorst (2001), “Nonuniform Sampling: Bandwidth and Aliasing,” in *Maximum Entropy and Bayesian Methods in Science and Engineering*, Joshua Rychert, Gary Erickson and C. Ray Smith *eds.*, pp. 1-28, American Institute of Physics, USA.
- [14] G. Larry Bretthorst, Christopher D. Kroenke, and Jeffrey J. Neil (2004), “Characterizing Water Diffusion In Fixed Baboon Brain,” in *Bayesian Inference And Maximum Entropy Methods In Science And Engineering*, Rainer Fischer, Roland Preuss and Udo von Toussaint *eds.*, AIP conference Proceedings, **735**, pp. 3-15.
- [15] G. Larry Bretthorst, William C. Hutton, Joel R. Garbow, and Joseph J.H. Ackerman (2005), “Exponential parameter estimation (in NMR) using Bayesian probability theory,” *Concepts in Magnetic Resonance*, 27A, Issue 2, pp. 55-63.
- [16] G. Larry Bretthorst, William C. Hutton, Joel R. Garbow, and Joseph J. H. Ackerman (2005), “Exponential model selection (in NMR) using Bayesian probability theory,” *Concepts in Magnetic Resonance*, 27A, Issue 2, pp. 64-72.
- [17] G. Larry Bretthorst, William C. Hutton, Joel R. Garbow, and Joseph J.H. Ackerman (2005), “How accurately can parameters from exponential models be estimated? A Bayesian view,” *Concepts in Magnetic Resonance*, 27A, Issue 2, pp. 73-83.
- [18] G. Larry Bretthorst, W. C. Hutton, J. R. Garbow, and Joseph J. H. Ackerman (2008), “High Dynamic Range MRS Time-Domain Signal Analysis,” *Magn. Reson. in Med.*, **62**, pp. 1026-1035.
- [19] V. Chandramouli, K. Ekberg, W. C. Schumann, S. C. Kalhan, J. Wahren, and B. R. Landau (1997), “Quantifying gluconeogenesis during fasting,” *American Journal of Physiology*, **273**, pp. H1209-H1215.
- [20] R. T. Cox (1961), “The Algebra of Probable Inference,” Johns Hopkins Univ. Press, Baltimore.
- [21] André d’Avignon, G. Larry Bretthorst, Marlyn Emerson Holtzer, and Alfred Holtzer (1998), “Site-Specific Thermodynamics and Kinetics of a Coiled-Coil Transition by Spin Inversion Transfer NMR,” *Biophysical Journal*, **74**, pp. 3190-3197.
- [22] André d’Avignon, G. Larry Bretthorst, Marlyn Emerson Holtzer, and Alfred Holtzer (1999), “Thermodynamics and Kinetics of a Folded-Folded Transition at Valine-9 of a GCN4-Like Leucine Zipper,” *Biophysical Journal*, **76**, pp. 2752-2759.
- [23] David Freedman, and Persi Diaconis (1981), “On the histogram as a density estimator:  $L_2$  theory,” *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, **57**, 4, pp. 453-476.
- [24] W. R. Gilks, S. Richardson, and D. J. Spiegelhalter (1996), “Markov Chain Monte Carlo in Practice,” Chapman & Hall, London.

- [25] Paul M. Goggans, and Ying Chi (2004), “Using Thermodynamic Integration to Calculate the Posterior Probability in Bayesian Model Selection Problems,” in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering: 23rd International Workshop*, **707**, pp. 59-66.
- [26] Marlyn Emerson Holtzer, G. Larry Bretthorst, D. André d’Avignon, Ruth Hogue Angelette, Lisa Mints, and Alfred Holtzer (2001), “Temperature Dependence of the Folding and Unfolding Kinetics of the GCN4 Leucine Lipper via  $^{13}\text{C}$  alpha-NMR,” *Biophysical Journal*, **80**, pp. 939-951.
- [27] E. T. Jaynes (1968), “Prior Probabilities,” *IEEE Transactions on Systems Science and Cybernetics*, SSC-4, pp. 227-241; reprinted in [30].
- [28] E. T. Jaynes (1978), “Where Do We Stand On Maximum Entropy?” in *The Maximum Entropy Formalism*, R. D. Levine and M. Tribus Eds., pp. 15-118, Cambridge: MIT Press, Reprinted in [30].
- [29] E. T. Jaynes (1980), “Marginalization and Prior Probabilities,” in *Bayesian Analysis in Econometrics and Statistics*, A. Zellner ed., North-Holland Publishing Company, Amsterdam; reprinted in [30].
- [30] E. T. Jaynes (1983), “Papers on Probability, Statistics and Statistical Physics,” a reprint collection, D. Reidel, Dordrecht the Netherlands; second edition Kluwer Academic Publishers, Dordrecht the Netherlands, 1989.
- [31] E. T. Jaynes (1957), “How Does the Brain do Plausible Reasoning?” unpublished Stanford University Microwave Laboratory Report No. 421; reprinted in *Maximum-Entropy and Bayesian Methods in Science and Engineering* **1**, pp. 1-24, G. J. Erickson and C. R. Smith Eds., 1988.
- [32] E. T. Jaynes (2003), “Probability Theory—The Logic of Science,” edited by G. Larry Bretthorst, Cambridge University Press, Cambridge UK.
- [33] Sir Harold Jeffreys (1939), “Theory of Probability,” Oxford Univ. Press, London; Later editions, 1948, 1961.
- [34] John G. Jones, Michael A. Solomon, Suzanne M. Cole, A. Dean Sherry, and Craig R. Malloy (2001) “An integrated  $^2\text{H}$  and  $^{13}\text{C}$  NMR study of gluconeogenesis and TCA cycle flux in humans,” *American Journal of Physiology, Endocrinology, and Metabolism*, **281**, pp. H848-H856.
- [35] John Kotyk, N. G. Hoffman, W. C. Hutton, G. Larry Bretthorst, and J. J. H. Ackerman (1992), “Comparison of Fourier and Bayesian Analysis of NMR Signals. I. Well-Separated Resonances (The Single-Frequency Case),” *J. Magn. Reson.*, **98**, pp. 483–500.
- [36] Pierre Simon Laplace (1814), “A Philosophical Essay on Probabilities,” John Wiley & Sons, London, Chapman & Hall, Limited 1902. Translated from the 6th edition by F. W. Truscott and F. L. Emory.
- [37] N. Lartillot, and H. Philippe (2006), “Computing Bayes Factors Using Thermodynamic Integration,” *Systematic Biology*, **55** (2), pp. 195-207.

- [38] D. Le Bihan, and E. Breton (1985), “Imagerie de diffusion in-vivo par rsonance,” Comptes rendus de l’Acadmie des Sciences (Paris), **301** (15), pp. 1109-1112.
- [39] N. R. Lomb (1976), “Least-Squares Frequency Analysis of Unevenly Spaced Data,” *Astrophysical and Space Science*, **39**, pp. 447-462.
- [40] T. J. Loredo (1990), “From Laplace To SN 1987A: Bayesian Inference In Astrophysics,” in *Maximum Entropy and Bayesian Methods*, P. F. Fougere (ed), Kluwer Academic Publishers, Dordrecht, The Netherlands.
- [41] Craig R. Malloy, A. Dean Sherry, and Mark Jeffrey (1988), “Evaluation of Carbon Flux and Substrate Selection through Alternate Pathways Involving the Citric Acid Cycle of the Heart by  $^{13}\text{C}$  NMR Spectroscopy,” *Journal of Biological Chemistry*, **263** (15), pp. 6964-6971.
- [42] Craig R. Malloy, Dean Sherry, and Mark Jeffrey (1990), “Analysis of tricarboxylic acid cycle of the heart using  $^{13}\text{C}$  isotope isomers,” *American Journal of Physiology*, **259**, pp. H987-H995.
- [43] Lawrence R. Mead and Nikos Papanicolaou, “Maximum entropy in the problem of moments,” *J. Math. Phys.* **25**, 2404–2417 (1984).
- [44] K. Merboldt, Wolfgang Hanicke, and Jens Frahm (1969), “Self-diffusion NMR imaging using stimulated echoes,” *Journal of Magnetic Resonance*, **64** (3), pp. 479-486.
- [45] Nicholas Metropolis, Arianna W. Rosenbluth, Marshall N. Rosenbluth, Augusta H. Teller, and Edward Teller (1953), “Equation of State Calculations by Fast Computing Machines,” *Journal of Chemical Physics*. The previous link is to the Americain Institute of Physics and if you do not have access to Science Sitations you many not be able to retrieve this paper.
- [46] Radford M. Neal (1993), “Probabilistic Inference Using Markov Chain Monte Carlo Methods,” technical report CRG-TR-93-1, Dept. of Computer Science, University of Toronto.
- [47] Jeffrey J. Neil, and G. Larry Bretthorst (1993), “On the Use of Bayesian Probability Theory for Analysis of Exponential Decay Data: An Example Taken from Intravoxel Incoherent Motion Experiments,” *Magn. Reson. in Med.*, **29**, pp. 642–647.
- [48] H. Nyquist (1924), “Certain Factors Affecting Telegraph Speed,” *Bell System Technical Journal*, **3**, pp. 324-346.
- [49] H. Nyquist (1928), “Certain Topics in Telegraph Transmission Theory,” *Transactions AIEE*, **3**, pp. 617-644.
- [50] William H. Press, Saul A. Teukolsky, William T. Vetterling and Brian P. Flannery (1992), “Numerical Recipes The Art of Scientific Computing Second Edition,” Cambridge University Press, Cambridge UK.
- [51] Emanuel Parzen (1962), “On Estimation of a Probability Density Function and Mode,” *Annals of Mathematical Statistics* **33**, 1065–1076
- [52] Karl Pearson (1895), “Contributions to the Mathematical Theory of Evolution. II. Skew Variation in Homogeneous Material,” *Phil. Trans. R. Soc. A* **186**, 343–326.

- [53] Murray Rosenblatt, "Remarks on Some Nonparametric Estimates of a Density Function," *Annals of Mathematical Statistics* **27**, 832–837 (1956).
- [54] Jeffery D. Scargle (1981), "Studies in Astronomical Time Series Analysis I. Random Process In The Time Domain," *Astrophysical Journal Supplement Series*, **45**, pp. 1-71.
- [55] Jeffery D. Scargle (1982), "Studies in Astronomical Time Series Analysis II. Statistical Aspects of Spectral Analysis of Unevenly Sampled Data," *Astrophysical Journal*, **263**, pp. 835-853.
- [56] Jeffery D. Scargle (1989), "Studies in Astronomical Time Series Analysis. III. Fourier Transforms, Autocorrelation Functions, and Cross-correlation Functions of Unevenly Spaced Data," *Astrophysical Journal*, **343**, pp. 874-887.
- [57] Arthur Schuster (1905), "The Periodogram and its Optical Analogy," *Proceedings of the Royal Society of London*, **77**, p. 136-140.
- [58] Claude E. Shannon (1948), "A Mathematical Theory of Communication," *Bell Syst. Tech. J.*, **27**, pp. 379-423.
- [59] John E. Shore, and Rodney W. Johnson (1981), "Properties of cross-entropy minimization," *IEEE Trans. on Information Theory*, **IT-27**, No. 4, pp. 472-482.
- [60] John E. Shore and Rodney W. Johnson (1980), "Axiomatic derivation of the principle of maximum entropy and the principle of minimum cross-entropy," *IEEE Trans. on Information Theory*, **IT-26** (1), pp. 26-37.
- [61] Devinderjit Sivia, and John Skilling (2006), "Data Analysis: A Bayesian Tutorial," Oxford University Press, USA.
- [62] Edward O. Stejskal and Tanner, J. E. (1965), "Spin Diffusion Measurements: Spin Echoes in the Presence of a Time-Dependent Field Gradient." *Journal of Chemical Physics*, **42** (1), pp. 288-292.
- [63] D. G. Taylor and Bushell, M. C. (1985), "The spatial mapping of translational diffusion coefficients by the NMR imaging technique," *Physics in Medicine and Biology*, **30** (4), pp. 345-349.
- [64] Myron Tribus (1969), "Rational Descriptions, Decisions and Designs," Pergamon Press, Oxford.
- [65] P. M. Woodward (1953), "Probability and Information Theory, with Applications to Radar," McGraw-Hill, N. Y. Second edition (1987); R. E. Krieger Pub. Co., Malabar, Florida.
- [66] Arnold Zellner (1971), "An Introduction to Bayesian Inference in Econometrics," John Wiley and Sons, New York.