Propagation in Ferrite-Filled Transversely Magnetized Waveguide*

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Summary—**A solution to the problem of propagation of higher modes in a transversely magnetized ferrite-filled rectangular waveguide has been found. The solutions to the problem are expressed in the form of four rigorous nonlinear algebraic equations which characterize the problem and are ready for numerical solution. The dependence of the fields in the direction of magnetization is the same as for the classical modes.**

WE SHALL consider the problem of propagation in a rectangular waveguide which is completely filled with ferrite and magnetized transversely to the direction of propagation.

This problem is becoming of more interest as lower loss ferrites are developed. As these very low-loss ferrites become available, a class of devices depending on the ability of a dc magnetic field to change the propagation constant within a waveguide will become practical. With the transverse field geometry, these devices will operate at low field values far from gyromagnetic resonance. They will be characterized by transverse fields which are distorted by the applied magnetic field. This will make this geometry useful in field displacement devices such as isolators and radiators.

We shall hence find the fields and propagation constants for the modes in this particular ferrite geometry. They will be characterized by parameters which continuously vary with increasing magnetic field from the classical TE and TM modes into a new set of modes having fields and propagation constants which are magnetically controllable.

Gamo[1] and Kales[2] have investigated the case of the longitudinally magnetized filled cylindrical waveguide. Van Trier[3] has solved the case of the TE_{10} mode in the transversely magnetized waveguide and found that the new mode is a TE mode with a distorted transverse field dependence. Mikaelyan[4] and recently Chevalier,

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[1] H. Gamo, "The Faraday rotation of waves in a circular waveguide," *J. Phys. Soc. Jap.*, vol. 8, p. 176; March, 1953.
[2] M. L. Kales, "Modes in waveguides containing ferrites," *J. Appl. Phys.* vol, 24, p. 609, May, 1953.
[3] A. A. Van Trier, Th. M., paper presented orally at meeting of Amer. Phys. Soc., Washington, D.C.; April, 1952.
[4] A. L. Mikaelyan, "Electromagnetic waves in a rectangular waveguide filled with a magnetized ferrite," *Doklady*, A.N. USSR, vol. 98, p. 941; October, 1954.

Kahan, and Polacco[5] have also worked on the problem.

The problem then, is that of propagation in an infinitely long rectangular waveguide shown in Fig. 1, which is filled with ferrite and transversely magnetized along the x direction. The solutions to the problem will be expressed in terms of four rigorous nonlinear algebraic equations which characterize the problem and are ready for numerical solution.

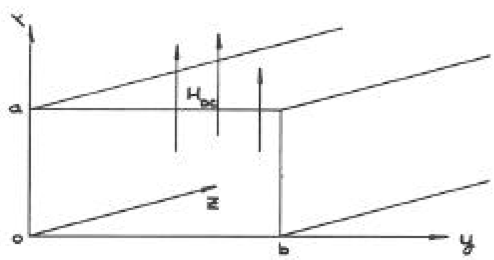

Fig. 1—Coordinate system.

Eqs. (1) and (2) are Maxwell's equations written in a form to show the tensor permeability. All field quantities vary as exp $(j\omega t-\gamma z)$.

$$\begin{pmatrix} 0 & \gamma & \frac{\partial}{\partial y} \\ -\gamma & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -j\omega\mu_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & -jK \\ 0 & jK & \mu \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 0 & \gamma & \frac{\partial}{\partial y} \\ -\gamma & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = j\omega\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{2}$$

The elements in the permeability tensor are known,[6] given the applied magnetic field and frequency. For zero applied field, μ becomes unity and K zero. It is important to note that the tensor properties of the ferrite are limited to the y-z plane, that is, the plane perpendicular to the applied field.

From Maxwell's equations we derive the relations for the transverse fields in terms of the longitudinal fields.

$$E_y = \frac{1}{k_{1x}^2}\left(j\omega\mu_0 \frac{\partial H_z}{\partial x} - \gamma \frac{\partial E_z}{\partial y}\right) \tag{3}$$

$$H_x = \frac{1}{k_{1x}^2}\left(j\omega\epsilon \frac{\partial E_z}{\partial y} - \gamma \frac{\partial H_z}{\partial x}\right) \tag{4}$$

$$E_x = \frac{-1}{k_{1y}^2}\left(j\omega\mu_0\mu \frac{\partial H_z}{\partial y} + \gamma \frac{\partial E_z}{\partial x} - \omega\mu_0 K\gamma H_z\right) \tag{5}$$

$$H_y = \frac{-1}{k_{1y}^2}\left(j\omega\epsilon \frac{\partial E_z}{\partial x} + \gamma \frac{\partial H_z}{\partial y} - jk^2 K H_z\right) \tag{6}$$

where

$$k_{1x}^2 = \gamma^2 + k^2 \qquad k^2 = \omega^2\epsilon\mu_0$$

$$k_{1y}^2 = \gamma^2 + k^2\mu$$

It is seen that these are of the usual form except that the E_x and H_y relations have extra terms proportional to H_z. Physically this is the rotational effect of the ferrite, that is, electrons in the ferrite driven in the z direction are caused to precess and generate a field in the perpendicular plane. We shall work with the E_z and H_z fields and the transverse fields can then be found from these equations.

The two differential equations which E_z and H_z must satisfy are obtained from Maxwell's equations.

$$\omega\epsilon\left[\frac{j\gamma X}{k_{1x}^2 k_{1y}^2}\frac{\partial^2}{\partial x \partial y} + \frac{K}{k_{1y}^2}\frac{\partial}{\partial x}\right]E_z + \left[\frac{1}{k_{1x}^2}\frac{\partial^2}{\partial x^2} + \frac{\mu}{k_{1y}^2}\frac{\partial^2}{\partial y^2} + \mu - \frac{k^2K^2}{k_{1y}^2}\right]H_z = 0 \tag{7}$$

$$\left[\frac{1}{k_{1y}^2}\frac{\partial^2}{\partial x^2} + \frac{1}{k_{1x}^2}\frac{\partial^2}{\partial y^2} + 1\right]E_z + \omega\mu_0\left[\frac{j\gamma X}{k_{1x}^2 k_{1y}^2}\frac{\partial^2}{\partial x \partial y} - \frac{K}{k_{1y}^2}\frac{\partial}{\partial x}\right]H_z = 0 \tag{8}$$

where $X=\mu-1$.

There are three interesting cases here. For zero applied field, the first term in the first equation and second term in the second equation, are zero since X and K are zero. The remaining expressions reduce to the usual forms for the classical TE and TM modes. Secondly if there is no variation of the fields in the x direction, then the TE_{no} modes found by Van Trier[3] having a distorted transverse dependence and a magnetically controllable propagation constant result. The third case is the general one, of all the other higher order modes.

The fact that the tensor properties are limited to the y-z plane suggests the form of the solutions shown in (9).

$$E_z = f(y)\sin\frac{m\pi x}{a}; \qquad H_z = g(y)\cos\frac{m\pi x}{a}. \tag{9}$$

[5] A. Chevalier, T. Kahan, and E. Polacco, "Propagation des ondes electromagnetiques dans un milieu gyromagnetique anisotrope, contenu dans un guide rectangulaire," *Compt. Rend. (Paris)*, vol. 240, pp. 1323–1324; March, 1955.

[6] C. L. Hogan, "The ferromagnetic Faraday effect at microwave frequencies and its applications," *Bell Sys. Tech. J.* vol. 31, pp. 1–31; January, 1952.

Hence the x dependence of the fields remains unaltered by the ferrite. Substituting this form of fields into the differential equations, the x dependence drops out and we are left with two second order linear differential equations in f and g. The determinantal equation for these two equations is

$$\left[\frac{\partial^4}{\partial y^4} + B\frac{\partial^2}{\partial y^2} + C\right]\begin{Bmatrix} f \\ g \end{Bmatrix} = 0 \tag{10}$$

where B and C depend on the propagation constant, frequency, and applied field. Thus the functions f and g may be represented as a sum of four independent trigonometric or exponential functions.

We will choose solutions consisting of products of two trigonometric functions each having a different argument, ry and qy.

$$\begin{Bmatrix} \sin \\ \cos \end{Bmatrix} ry \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} qy.$$

This particular form is suggested by the requirement that the fields reduce to the usual TE and TM modes for the case of zero applied field. Hence we would expect that r would go to $n\pi/b$ and q to zero for zero applied field. As an example, a plot for small values of q of $\sin ry \cos qy$ is shown in Fig. 2. It is seen that fields described by this function are distorted towards one side of the waveguide as the magnetic field is applied. As in the case of the TE_{no} modes this may result in a Poynting vector which on one side of the guide is opposite to the direction of propagation. This may be thought of as a uniform Poynting vector with a superimposed circulating energy.

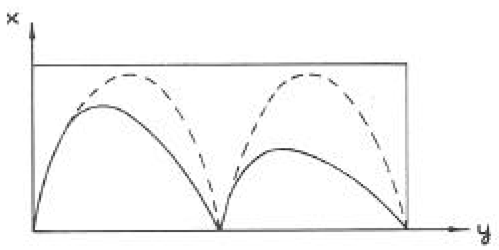

Fig. 2—Distortion of transverse fields described by $\sin ry \cos qy$ for small q.

Substituting any of these four solutions in (10) yields two relations which must be satisfied by the unknowns r and q.

$$\gamma^2 = \left[-k^2\left(\frac{1+\mu_e}{2}\right) + \left(\frac{1+\mu}{2\mu}\right)\left(\frac{m\pi}{a}\right)^2\right] + r^2 + q^2 \tag{11}$$

$$r^2q^2 = \frac{k^4(\mu_e - 1)^2}{16} + \left(\frac{m\pi}{a}\right)^2 \frac{k^2}{2}\left[1 - \left(\frac{1+\mu}{4\mu}\right)(1+\mu_e)\right] + \left(\frac{X}{4\mu}\right)^2\left(\frac{m\pi}{a}\right)^4 \equiv F^2 \tag{12}$$

and

$$\mu_e = \frac{\mu^2 - K^2}{\mu}.$$

For zero applied field the propagation constant shown in (11) goes to the usual form since the μ and μ_e become unity, r goes to $(n\pi/b)$ and q vanishes, as we postulated when choosing the form of the solutions. A further relation between r and q, (12), states that their product squared is a function only of the applied magnetic field and frequency. Hence q is known as a function of r, and only r must be determined in order to find the propagation constant.

The E_z and H_z fields are shown in (13) and (14):

$$E_z = R[S \sin ry \sin qy + T \cos ry \sin qy + \sin ry \cos qy] \sin \frac{m\pi x}{a} \tag{13}$$

$$H_z = L[M \sin ry \cos qy + N \cos ry \sin qy + P \sin ry \sin qy + \cos ry \cos qy] \cos \frac{m\pi x}{a}. \tag{14}$$

The boundary condition on the E_z field at $y=0$ required the cos cos term to be identically zero. At $y=b$ the boundary conditions require the quantity in the bracket in (13) to be zero.

The H_z field in (14) must satisfy the boundary condition specified by

$$\left(\frac{\partial H_z}{\partial y} + \frac{j\gamma K H_z}{\mu}\right)_{y=0,b} = 0. \tag{15}$$

This magnetic boundary condition is most easily found by requiring that E_z be zero at the walls. Note that this boundary condition is different from the usual in that an extra term is present. Substituting (14) into (15) yields two equations which along with the one equation from the E_z fields gives 3 equations in 6 unknown amplitudes and the quantity r. We hence need 4 more relations which are found by substituting the E_z and H_z fields into one of the original longitudinal differential equations. This can be manipulated into a set of 4 nonlinear algebraic equations in four unknowns.

$$\frac{G - J \tan rb \cot qb}{J - G \tan rb \cot qb} = \frac{G \tan qb - \phi(-1, P, -M)}{J \tan qb - \phi(P, -1, -N)} \tag{16}$$

$$\left(-r + Pq + M\frac{j\gamma K}{\mu}\right)\sin rb \cos qb + \left(Pr - q + N\frac{j\gamma K}{\mu}\right)\cos rb \sin qb + \left(-Nr - Mq + P\frac{j\gamma K}{\mu}\right)\sin rb \sin qb = 0 \tag{17}$$

$$\phi(N, M, P) = GK \tag{18}$$

$$\mu(Mr + Nq) = -j\gamma K \tag{19}$$

where

$$G = k_{1y}^2\left(k_{1z}^2 - \frac{B}{2}\right) - \left(\frac{m\pi}{a}\right)^2 k_{1x}^2$$

$$J = -2Fk_{1y}^2$$

$$\phi(u, v, w) = 2F\mu[j\gamma X(ur + vq) + Kk_{1x}^2 w].$$

The unknowns here are the three magnetic field amplitudes and r. The electric amplitudes are known in terms of these parameters. The theory leading to these equations has been rigorous and they are now ready for solution by numerical methods or by approximation techniques.

It can be shown that there are no pure TE or pure TM modes allowed in the magnetized case. A similar result was found by Gamo and Kales in their treatment of the longitudinally magnetized cylindrical waveguide. This is physically reasonable since the transverse magnetic fields for the TM modes now generate longitudinal fields through the rotational nature of the ferrite and thus TM modes would not be expected. Maxwell's equations permite TE modes only for modes with zero x dependence and these are Van Trier's TE_{no} modes.

In conclusion we have derived a set of four nonlinear equations whose solution determines a rigorous solution to the problem of propagation in a transversely magnetized ferrite-filled waveguide. The fields can be expressed in the form of products of two trigonometric functions with arguments which are asymptotic to $n\pi y/b$ and 0 in the limit of zero applied field. The product of these arguments is dependent only on the magnetic field and frequency.