7/8/1996.

NOTE ON THERMAL HEATING EFFICIENCY

E. T. Jaynes
Wayman Crow Professor of Physics
Washington University
St. Louis MO 63130

Abstract: Kelvin showed the maximum efficiency with which heat can be converted into work; but there is a dual theorem about the maximum efficiency with which heat at one temperature can be converted into heat at another temperature. It has some surprising implications, in particular that the efficiency with which we heat our buildings could in principle be improved by a large factor. This long known – but still little known – fact is of current pedagogical interest and practical importance.

1. INTRODUCTION

For over 200 years the University of Glasgow, Scotland, has played a uniquely important role in the development of thermodynamics. There the distinction between temperature as a measure of *intensity* of something; and heat as a *quantity* of something, was first seen clearly by Joseph Black, about 1760. This knowledge contributed to the work of his colleague, James Watt, in the first practical means of converting heat into work. Then Carnot and others tried to find the maximum theoretical efficiency of this conversion, but the one who finally succeeded was Wm. Thomson (later Lord Kelvin) at the University of Glasgow.

Recently an addition to this was made, which is not only of theoretical interest as representing in a sense the completion of the logical structure of classical thermodynamics; it has immediate practical implications. Yet the principle is hardly new; it is such a simple and immediate consequence of Thomson's work that it must have been known to Thomson in 1870. Today it cannot be really unknown to anyone familiar with the theory of heat pumps. But to the best of our knowledge it has not yet appeared in any physics textbook, stated in a form where it is seen as logically independent of Carnot engines, and forming the natural dual theorem to the one on efficiency of Carnot engines. It seems appropriate that this way of looking at the result was finally pointed out by Robert S. Silver (1981), the James Watt Professor (now emeritus) of the University of Glasgow.

In Section 2 we give the almost trivial derivation, and in Section 3 we point out its practical implications by numerical examples. Since a large part of the world's energy resources are actually used for heating rather than production of work, these implications are not trivial. Section 4 points out another surprising application, and in the final Section 5 we speculate on possible nonequilibrium generalizations.

2. THEORETICAL DERIVATION

We have a source of heat Q_2 which is available at Kelvin temperature T_2. By this we mean, as was stressed long ago by J. Willard Gibbs (1886), that the source is capable of delivering that heat to a heat reservoir which is at temperature T_2; and T_2 is the highest temperature to which it can deliver that heat. If there is available a cold reservoir at temperature $T_1 < T_2$, then according to classical thermodynamics we may exploit this temperature difference to obtain work W. Applying the first and second laws: $W = Q_2 - Q_1, \quad Q_1/T_1 \geq Q_2/T_2$ and solving these for W and Q_1 we have

$$W \leq Q_2 \left(1 - \frac{T_1}{T_2}\right), \qquad Q_1 \geq Q_2 \frac{T_1}{T_2} \tag{1}$$

with equality if and only if the engine is reversible. In the latter case the "wasted energy"

$$Q_1(\text{Carnot}) = Q_2 \frac{T_1}{T_2} \tag{2}$$

is delivered as heat to the reservoir at temperature T_1. This is the standard result.

But now suppose that our objective is not to produce work, but to deliver the maximum possible heat to that lower temperature reservoir. This is the conversion problem faced in every home, where one has heat from a gas, oil, wood, or coal flame but wants heat at room temperature. At present, we simply allow the primary heat Q_2 to degrade itself directly to the lower temperature T_1 by passing through ducts, radiators, etc. Thus we obtain, at best (*i.e.*, neglecting heat loss through chimneys), the amount of heat $Q_1(\text{direct}) = Q_2$. But this is an irreversible process, since there is a net entropy increase $\Delta S = Q_2/T_1 - Q_2/T_2 > 0$ indicating that something has been wasted, and we can do better. The first and second laws imply that, not only in conversion of heat to work, but also in conversion of heat to heat, the maximum efficiency will be attained if we can carry out the process reversibly.

Suppose we have an ambient heat reservoir (the outside world) at temperature $T_0 < T_1$ and we use a perfect Carnot engine to obtain the heat $Q_1(\text{Carnot})$. Then we still have the work W available, which we can use to drive a heat pump between T_0 and T_1, yielding the additional heat

$$Q_1(\text{pump}) = \frac{T_1 W}{T_1 - T_0} . \tag{3}$$

Combining (2) and (3), we have now obtained the total heat

$$Q_1 \ = \ Q_1(\text{Carnot}) + Q_1(\text{pump}) \ = \ Q_2 \frac{T_1}{T_2} \frac{T_2 - T_0}{T_1 - T_0} \tag{4}$$

and there is always a net gain, since Q_1 is always greater than Q_2 whenever $T_0 < T_1 < T_2$. But while we know that a reversible Carnot engine delivers the maximum attainable work, this argument does not make it obvious whether (4) is the maximum attainable heat.

Now from a theoretical standpoint it is more general and more elegant to apply the first and second laws directly to this process, as we did in (1). Since some heat Q_0 is removed from the outside reservoir, we must have

$$Q_1 = Q_0 + Q_2, \qquad \frac{Q_1}{T_1} \geq \frac{Q_0}{T_0} + \frac{Q_2}{T_2} . \tag{5}$$

Solving these equations for Q_1 and Q_0, we have

$$Q_1 \leq Q_2 \frac{T_1}{T_2} \frac{T_2 - T_0}{T_1 - T_0}, \qquad Q_0 \leq Q_2 \frac{T_0}{T_2} \frac{T_2 - T_1}{T_1 - T_0} \tag{6}$$

with equality if and only if the process is reversible. Thus we obtain automatically the same result (4), plus the statement that it is the *maximum attainable* heating, without invoking Carnot engines at all. It is in this simple argument that the main theoretical and pedagogical interest of this discussion lies.

3. PRACTICAL IMPLICATIONS

Consider heating from a primary temperature $T_2 = 1000K$ to room temperature, $T_1 = 25C = 298K$, with an outside temperature $T_0 = 0C = 273K$. Comparing our ideal Q_1 with the present maximum Q_2, we have from (7), the gain factor

$$G \equiv \frac{Q_1}{Q_2} = \frac{1 - .273}{1 - .916} = 8.66 \tag{7}$$

This seems at first glance quite startling; if we take into account that we are at present far from getting even Q_2 because of heat loss up chimneys, the conclusion is that it is in principle possible to heat our homes with an order of magnitude less fuel than we are now consuming.

A better idea of the numerical improvement allowed by the second law is given in Fig. 1, where we give contours of constant gain $G \equiv Q_1/Q_2$ in the (T_0, T_2) plane for $T_1 = 25\ ^o$C, room temperature. Even in cold climates, average gains of the order of 5 are indicated.

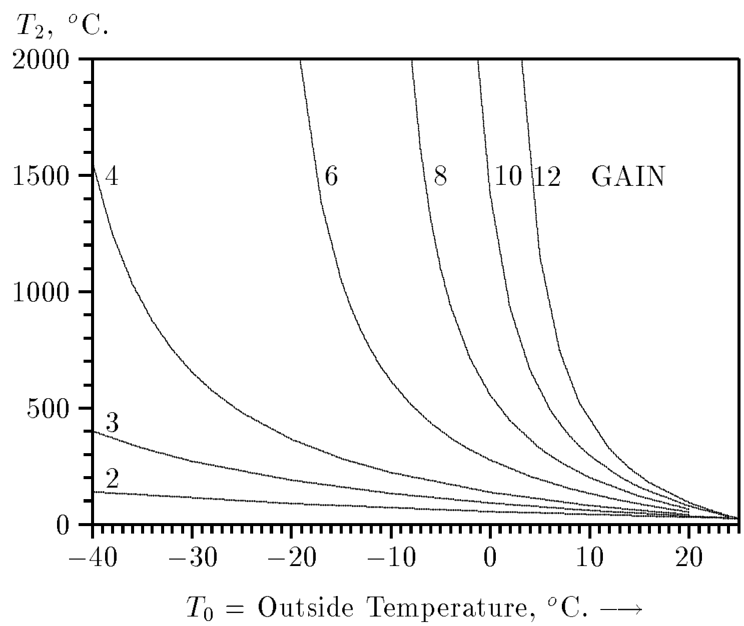

Figure 1. Contours of Constant Gain in the $(T_0,\ T_2)$ Plane, for $T_1 = 25\,^oC$.

The reason for this high efficiency is that T_0 and T_1 are not very different on the Kelvin scale. With the values of inside and outside temperature assumed in (7), one Joule of work will pump

$$T_0/(T_1 - T_0) = 10.9 \tag{8}$$

Joules of heat from the outside world, and deliver 11.9 joules to the inside. Unfortunately, presently available heat pumps are far from realizing this theoretical efficiency. Silver (1981) notes that if present engines realize only half of the theoretical efficiency, then the heat pump component of Q_1 will be only a quarter of our calculated value.

Evidently, the development of heat pumps that approach the theoretical efficiency for small temperature differences would be of very great economic importance, and no physical law stands in the way of realizing them. It is only a matter of the ingenuity of inventors; and the one who succeeds will be one of the world's great benefactors. We suspect that the successful technology will avoid the crude mechanical pumps of our present realizations, perhaps depending on thermoelectric or electrochemical means that avoid all mechanical moving parts, although perhaps with circulating fluids.

4. FREE OVENS FOR ESKIMOS

Note that the derivation of (6) is general in that it holds for any exchange of heat between three reservoirs whatever the relative temperatures and the signs of the Q_i, although the arrangement of Carnot engines envisaged in our derivation of (4) would no longer apply. But this seems to contradict one common statement of the second law, attributed to Kelvin, that "It is impossible for heat to flow of itself from a cold reservoir to a hotter one". The statement actually made by Kelvin is that it is impossible to do this *without leaving changes in external bodies.* Eq. (6) demonstrates the need for this qualification; for it is quite possible for heat to flow spontaneously from room temperature T_1 to a higher temperature T_2, if there is at the same time a compensating flow to a lower temperature T_0.

Suppose then that we want to heat an oven at the standard cooking temperature of $T_2 = 400F = 204C = 477K$, using heat extracted from the air of a kitchen at room temperature $T_1 = 25C = 298K$. Our equations use the sign convention that Q_1 is heat *delivered to* the reservoir at T_1, while Q_0 and Q_2 represent heat *extracted from* those at T_0, T_2. Therefore Q_0, Q_1, Q_2 are now all negative, so $(-Q_1)$ is the heat extracted from the room and $(-Q_2)$ is the resulting heat delivered to the oven; but Eqs. (6) still hold. Writing the first as

$$(-Q_2) \leq (-Q_1)\,\frac{1 - T_0/T_1}{1 - T_0/T_2}, \tag{9}$$

we see that the maximum heat that can be delivered to the oven is less than that extracted from the room, but if the outside temperature T_0 is low enough, the efficiency can be quite high; unlike room heating, oven heating becomes more efficient as the outside temperature is lowered.

Indeed, we have only to run a Carnot engine between T_1 and T_0 extracting the work $W = (-Q_1)(1 - T_0/T_1)$, then use that to run a heat pump between T_0 and T_2, which delivers the heat $(-Q_2) = W/(1 - T_0/T_2)$, in agreement with (9). If the outside temperature T_0 is $-40F = -40C = 233K$ then according to (9), 1000 calories of heat removed from the room can deliver 426 calories to the oven. If this leaks back eventually to re–heat the room, it might appear that the "cost" of running the oven was not the 1000 calories removed from the room, but only the 574 calories lost to the outside.

But this leaking back is again an irreversible process in which something is wasted, and we can do better. If the oven is well insulated, then when we are done with it the heat $(-Q_2)$ is still in it, so we have only to run those Carnot engines backwards, obtaining the work $W = 426\,(1 - T_0/T_2)$ from which the heat pump can return the heat $W/(1 - T_0/T_1) = 1000$ calories to the room, completely restoring the *status quo.* The second law allows us to operate an oven, at whatever temperature we please, at zero cost, the outside reservoir T_0 serving only as a temporary repository for the entropy that must be disposed of in heating the oven.

Unfortunately, the second law will not allow us to supply our cooling needs as easily; it offers free (that is, zero operating cost) ovens to eskimos, but not free air–conditioning to hottentots because they have no lower temperature reservoir to take up that entropy.

5. SPECULATIONS

How much generality and finality do the above results have? As we stressed before (Jaynes, 1965), in classical thermodynamics the notions of temperature and entropy are defined only for states of thermal equilibrium; therefore the conventional second law that we considered above refers only to the net result of processes that begin and end in states of thermal equilibrium.

Then classical thermodynamics does not in itself prohibit still more efficient engines, *if they operate in nonequilibrium conditions*; it is simply silent on that question. Indeed, the surprisingly high observed efficiency of animal muscles, which operate in a nonequilibrium environment, has been thought by some to be such a realized violation of the second law.

Many have speculated about the possibility of non–biological engines that violate the second law. The more careful writers have refrained from claiming that they are absolutely impossible in principle. The Maxwell Demon, which is able to operate on a system directly at the microscopic level, is the most familiar example; but Max Planck (1917) also noted that we expect "to make a most serviceable application" of any phenomenon that is found to deviate from the second law, and considered it a good policy to remain alert, on the lookout for such things.

Presumably, if fundamental limitations on conversion efficiency in nonequilibrium conditions exist, they will come instead from statistical mechanics; but in our opinion all existing attempts to show this contain logical loopholes, and no absolutely convincing arguments of this nature have been produced. We feel, as did Maxwell and Planck, that from the standpoint of logical demonstration this is still an open question; dogmatic pronouncements on either side are premature.

However, a nonequilibrium generalization of the second law, that in essence goes back to Boltzmann, does place definite restrictions on what can be accomplished *reproducibly* when our technology, unlike the Maxwell Demon, is without knowledge of the microstate and is able to operate only at the macroscopic level. Any macrostate M, equilibrium or nonequilibrium, represents a certain phase volume $W(M)$ occupied by all microstates compatible with M. A reproducible process must work for all of those microstates; so it is a direct consequence of Liouville's theorem that the entropy $S = k \log W$ cannot decrease in a reproducible macroscopic process $(M_1 \rightarrow M_2)$ that takes place between such states. The maximum efficiency of a reproducible macroprocess is attained when the phase volume *of those degrees of freedom that actually take part in the interactions* is the same for the initial and final macrostates: $W(M_1) = W(M_2)$.

Using this fact, we have shown (Jaynes, 1989) that the high efficiency of animal muscles may be predicted from two data: the heat of reaction 0.43 ev of hydrolysis of the ATP molecule and the value $37C$ of body temperature. Presumably, similar efficiencies are realizable *in vitro*, using systems that are never in thermal equilibrium. However, these phenomena do not really violate the principles explained by Maxwell, Gibbs, Planck, and Einstein long ago; they represent only the recognition that reversible operation need not be slow. In a nonequilibrium environment, maximum efficiency may require the reversible entropy–preserving process to be fast on the molecular time scale, so that the useful work is done before the inevitable final thermalization can take place. This, we suggest, is the secret of the high efficiency of muscles. The field now seems wide open for new and important advances.

REFERENCES

Gibbs, J. W. (1886), "Electrochemical Thermodynamics", Report Brit. Assoc. Adv. Sci. 388; reprinted in *The Scientific Papers of J. Willard Gibbs*, Longmans, Green & Co., (1906) and by Dover Publications, Inc., New York (1961); pp 406-412.

Jaynes, E. T. (1965), "Gibbs *vs.* Boltzmann Entropies", Am. Jour. Phys. **33**, 391.

Jaynes, E. T. (1989), "Clearing up Mysteries – the Original Goal", in *Maximum Entropy and Bayesian Methods*, J. Skilling, Editor, Kluwer Publishing Co., Holland, pp. 1–27.

Planck, Max (1917), *Treatise on Thermodynamics*, 5'th Edition, Longmans, Green, & Co., London; undated reprint by Dover Publications, Inc., New York; Part III, Chapter II.

Silver, R. S. (1981), "Reflexions sur la puissance *chaleurique* du feu", Heat Recovery Systems, **1**, 205–207.